

Thermal/quantum effects and induced superstring cosmologies*

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Abstract

We consider classical superstring theories on flat four dimensional space-times, and where $N = 4$ or $N = 2$ supersymmetry is spontaneously broken. We obtain the thermal and quantum corrections at the string one-loop level and show that the back-reaction on the space-time metric induces a cosmological evolution. We concentrate on heterotic string models obtained by compactification on a T^6 torus and on T^6/\mathbb{Z}_2 orbifolds. The temperature T and the supersymmetry breaking scale M are generated via the Scherk-Schwarz mechanism on the Euclidean time cycle and on an internal spatial cycle respectively. The effective field theory corresponds to a no-scale supergravity, where the corresponding no-scale modulus controls the Susy-breaking scale. The classical flatness of this modulus is lifted by an effective thermal potential, given by the free energy. The gravitational field equations admit solutions where M , T and the inverse scale factor $1/a$ of the universe remain proportional. In particular the ratio M/T is fixed during the time evolution. The induced cosmology is governed by a Friedmann-Hubble equation involving an effective radiation term $\sim 1/a^4$ and an effective curvature term $\sim 1/a^2$, whose coefficients are functions of the complex structure ratio M/T .

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1 Introduction

It is important to develop a string theoretic framework for studying cosmology. The ultimate goal of this task is to determine whether string theory can describe basic features of our Universe. Despite considerable effort towards this direction over the last few years (see for example [1] – [10]), still very little is known about the dynamics of string theory in time-dependent, cosmological settings. The purpose of this work is to provide a new class of non-trivial string theory cosmological solutions, where some of the difficult issues can be explored and analyzed concretely.

At the classical string level, it seems difficult to obtain exact cosmological solutions [8]. Indeed, after extensive studies in the framework of superstring compactifications (with or without fluxes), the obtained results appear to be unsuitable for cosmology. In most cases, the classical ground states correspond to static Anti-de Sitter or flat backgrounds but not to cosmological ones. The same situation appears to be true in the effective supergravity theories. Naively, the results obtained in this direction lead to the conclusion that cosmological ground states are unlikely to be found in superstring theory.

From our viewpoint this conclusion cannot be correct for two reasons:

- The first follows from the fact that already exact (to all orders in α') cosmological solutions exist, which are described by a two dimensional worldsheet conformal field theory based on a gauged Wess-Zumino-Witten model at negative level $-|k|$: $\frac{SL(2,R)_{-|k|}}{U(1)} \times M$, [2–4].
- The second is that quantum and thermal corrections are neglected in the classical string/supergravity regime.

The first class of stringy cosmological models was studied recently in [5], where it was shown how to define a normalizable wave-function for this class of backgrounds, realizing the Hartle-Hawking no-boundary proposal [11] in string theory. Explicit calculable examples were given for small values of the level $|k|$. As it was shown in [5], these models are intrinsically thermal with a temperature below but still close to the Hagedorn temperature. The disadvantage of small level $|k|$, however, is the absence of a semi-classical limit with $|k|$ arbitrarily large, which prevents us from obtaining a clean geometrical picture and studying issues such as back-reaction and particle production in a straightforward way.

Another direction consists of studying the “quantum and thermal cosmological solutions,”

which are generated dynamically at the quantum level of string theory [6,12]. Although this study looks to be hopeless and out of any systematic control, it turns out that in certain cases the quantum and thermal corrections are under control thanks to the special structure of the underlying effective supergravity theory in its spontaneously broken supersymmetric phase. An effective field theory study has already been initiated in [6,12]. (See also [13,14].)

In order to see how cosmological solutions arise naturally in this context, consider the case of a supersymmetric flat string background. At finite temperature the thermal fluctuations produce a non-zero energy density that is calculable perturbatively at the full string level. The back-reaction on the space-time metric and on certain of the moduli fields gives rise to a specific cosmological evolution. For temperatures below the Hagedorn temperature, the evolution of the universe is known to be radiation dominated. (See for instance [15,16] for some earlier work in this case and [16] for a review on string gas cosmology.)

More interesting cases are those where space-time supersymmetry is spontaneously broken at the string level either by geometrical [17] or non-geometrical fluxes. In the case where the geometrical fluxes are generated via freely acting orbifolds [18] – [23], the stringy quantum corrections are under control in a very similar way as the thermal ones. The back-reaction of the quantum and thermal corrections on the space-time metric and the moduli fields results in different kinds of cosmologies depending on the initial amount of supersymmetry ($N = 4, N = 2, N = 1$).

In this work we restrict attention to four-dimensional backgrounds with initial $N = 4$ or $N = 2$ space-time supersymmetry, obtained by toroidal compactification of the heterotic superstring on T^6 and T^6/\mathbb{Z}_2 -orbifolds. The spontaneous breaking of supersymmetry is implemented via freely acting orbifolds (as in [18] – [23]). The quantum and thermal corrections are determined simultaneously by considering the Euclidean version of the model where all coordinates are compactified: $S_T^1 \times T^3$ (for the four-dimensional space-time part) $\times M^6$ (for the internal manifold). Apart from being interesting in their own right, these examples may give us useful hints on how to handle the phenomenologically more relevant $N = 1$ cases. The $N = 1$ cases will be studied elsewhere.

The thermal corrections are implemented by introducing a coupling of the space-time fermion number Q_F to the string momentum and winding numbers associated to the Euclidean time cycle S_T^1 . The breaking of supersymmetry is generated by a similar coupling of an internal

R -symmetry charge Q_R to the momentum and winding numbers associated to an internal spatial cycle S_M^1 , e.g. the X_5 coordinate cycle.

We stress here, that the thermal and supersymmetry breaking couplings correspond to string theoretic generalizations of Scherk-Schwarz compactifications. Two very special mass scales appear both associated with the breaking of supersymmetry. These are the temperature scale $T \sim 1/(2\pi R_0)$ and the supersymmetry breaking scale $M \sim 1/(2\pi R_5)$, with R_0 and R_5 the radii of the Euclidean time cycle, S_T^1 , and of the internal spatial cycle, S_M^1 , respectively. The initially degenerate mass levels of bosons and fermions split by an amount proportional to T or M , according to the charges Q_F and Q_R . This mass splitting is the signal of supersymmetry breaking and gives rise to a non-trivial free energy density, which incorporates simultaneously the thermal corrections and quantum corrections due to the supersymmetry breaking boundary conditions along the spatial cycle S_M^1 .

At weak coupling, the free energy density can be obtained from the one-loop Euclidean string partition function [20] – [22]. The perturbative string amplitudes are free of the usual ultraviolet ambiguities that plague a field theoretic approach towards quantum gravity and cosmology. For large enough R_0 , R_5 , the Euclidean system is also free of tachyons – the presence of tachyons would correspond to infrared instabilities, driving the system towards a phase transition [21] – [23]. Therefore, the corresponding energy density and pressure can be determined unambiguously, and we can use them as sources in Einstein’s equations to obtain non-trivial cosmological solutions. This perturbative approach breaks down near the initial space-like singularity. We speculate whether this breakdown of perturbation theory can be associated with an early universe phase transition.

The paper is organized as follows. Section 2 is mainly a review, where we also fix most of our notations and conventions. We first consider the four-dimensional heterotic string models at finite temperature. We obtain the one-loop thermal partition function at the full string level, and then we discuss the effective field theory limit at large radius R_0 . We also review the analogous computation of the one loop string partition function at zero temperature and in the case where Susy-breaking boundary conditions are placed along the internal spatial cycle S_M^1 , [18] – [23]. In the large radius limit, the Einstein frame effective potential is proportional to the fourth power of the gravitino mass scale, and it can be positive or negative depending on the choice of the Susy-breaking operator Q_R .

In section 3, we consider the case where thermal and quantum corrections due to the supersymmetry breaking are present simultaneously. For the simplest choice $Q_R = Q_F$, the corresponding one-loop string partition function is invariant under the $T \leftrightarrow M$ exchange, manifesting the underlying temperature/gravitino mass scale duality of the models. This duality is broken by the other allowable choices for the Susy-breaking operator Q_R , which we classify for both the $N = 4$ and the $N = 2$ orbifold cases.

In the large radii R_0, R_5 limit, the pressure consists of two pieces: the purely thermal part which scales as $n_T^* T^4$, with the coefficient n_T^* being the number of all massless boson/fermion pairs in the initially supersymmetric theory, and another potential-like piece which scales as $n_V^* M^4$ and with the coefficient n_V^* being positive or negative depending on the choice of the operator Q_R . In both pieces, the rest of the dependence on the scales T and M can be expressed neatly in terms of non-holomorphic Eisenstein series of order $5/2$ whose variable is the complex structure-like ratio M/T . In addition, we incorporate the effects of small, continuous Wilson line deformations in our computation. Wilson lines along any of the internal spatial cycles, other than S_M^1 , introduce new mass scales, and pieces proportional to $\sim T^2$ and $\sim M^2$ arise in the effective thermodynamic quantities.

In section 4 we present our ansatz for the induced cosmological solutions. These are homogeneous and isotropic cosmologies for which the Susy-breaking scales T and M as well as the inverse of the scale factor $1/a$ evolve the same way in time, and so the ratio of any two of these quantities is constant. The form of this ansatz is dictated by the scaling properties of the effective energy density and pressure. The compatibility of the gravitational field equations with the equation of motion of the scalar modulus controlling the size of the gravitino mass scale fixes the ratio M/T . By solving the compatibility equations numerically, we find that in the absence of Wilson lines along S_M^1 , non-trivial four dimensional solutions exist when n_V^* is negative and the ratio $|n_V^*|/n_T^*$ is small enough. These conditions are satisfied by various models we describe explicitly in the paper. When we include Wilson lines along S_M^1 , the value of the ratio M/T for some of the solutions can be large or small, and so we can have models with a hierarchy for the scales M and T .

Having solved the compatibility equations, the time-dependence of the system is governed solely by the familiar Friedmann-Hubble equation. There is a radiation term, c_r/a^4 , whose coefficient c_r is positive in our examples. An effective curvature term, $-\hat{k}/a^2$, can be gen-

erated by turning on Wilson line deformations. The sign of \hat{k} can be a priori positive or negative, depending on the model. When we turn on the kinetic terms of some of the extra flat moduli, we generate an additional term that scales as c_m/a^6 (with c_m positive).

In section 5, we solve the Friedmann-Hubble equation for the various possible cases, and we elaborate on the properties of the cosmological solutions:

- When $c_r > 0$, we have standard hot big bang cosmologies with an intermediate radiation dominated era. The late time behavior is governed by the spatial curvature of the models.
- We also consider a priori possible exotic models characterized by $c_r < 0$. A big bang occurs when $c_m > 0$. The cosmological evolution always ends with a big crunch when $\hat{k} \geq 0$. The case $\hat{k} < 0$ however is more interesting. It involves either a first or second order phase transition between the big bang cosmology and a linearly expanding universe. The first case corresponds to a tunneling effect involving a gravitational instanton, while the transition is smooth in the second case. If the first order transition does not occur, the universe ends in a big crunch.

We finish with our conclusions and directions for future research.

2 Thermal and quantum corrections in heterotic backgrounds

Our starting point is the class of four dimensional string backgrounds obtained by toroidal compactification of the heterotic string on T^6 and T^6/\mathbb{Z}_2 orbifolds. Initially the amount of space-time supersymmetry is $N_4 = 4$ for the case of compactification on the T^6 torus and $N_4 = 2$ for the orbifold compactifications, and the four dimensional space-time metric is flat. Space-time supersymmetry is then spontaneously broken by introducing Scherk-Schwarz boundary conditions on an internal spatial cycle and/or by thermal corrections. Due to the supersymmetry breaking, the one-loop string partition function is non-vanishing, giving rise to an effective potential. Our aim is to determine the back-reaction on the initially flat metric and moduli fields.

At the one-loop level, the four dimensional string frame effective action is given by

$$S = \int d^4x \sqrt{-\det g} \left(e^{-2\phi} \left(\frac{1}{2} R + 2\partial_\mu \phi \partial^\mu \phi + \dots \right) - \mathcal{V}_{\text{String}} \right), \quad (2.1)$$

where ϕ is the 4d dilaton field and the ellipses stand for the kinetic terms of other moduli

fields (to be specified later). At zero temperature, the effective potential $\mathcal{V}_{\text{String}}$ can be obtained from the one-loop Euclidean string partition function as follows:

$$\frac{Z}{V_4} = -\mathcal{V}_{\text{String}}, \quad (2.2)$$

with V_4 the $4d$ Euclidean volume. The absence of a dilaton factor multiplying the potential term in the action is due to the fact that this arises at the one loop level.

At finite temperature, the one-loop Euclidean partition function determines the free energy density and pressure to this order

$$\frac{Z}{V_4} = -\mathcal{F}_{\text{String}} = P_{\text{String}}. \quad (2.3)$$

The subscript indicates that these densities are defined with respect to the string frame metric. The relevant Euclidean amplitude incorporates simultaneously the thermal corrections and quantum corrections which arise from the spontaneous breaking of supersymmetry and which are present even at zero temperature.

In order to determine the back-reaction of the (thermal and/or) quantum corrections, it is convenient to work in the Einstein frame where there is no mixing between the metric and the dilaton kinetic terms. We define as usual the complex field S ,

$$S = e^{-2\phi} + i\chi, \quad (2.4)$$

where χ is the axion field. Then after the Einstein rescaling of the metric, the one loop effective action becomes:

$$S = \int d^4x \sqrt{-\det g} \left[\frac{1}{2} R - g^{\mu\nu} K_{I\bar{J}} \partial_\mu \Phi_I \partial_\nu \bar{\Phi}_{\bar{J}} - \frac{1}{s^2} \mathcal{V}_{\text{String}}(\Phi_I, \bar{\Phi}_{\bar{I}}) \right], \quad (2.5)$$

where $K_{i\bar{j}}$ is the metric on the scalar field manifold $\{\Phi_I\}$, which is parameterized by various compactification moduli including the field S . This manifold includes also the main moduli fields T_I, U_I , $I = 1, 2, 3$, which are the volume and complex structure moduli of the three internal 2-cycles respectively. We notice that in the Einstein frame the effective potential, $\mathcal{V}_{\text{String}}$, is rescaled by a factor $1/s^2$, where $s = \text{Re}(S) = e^{-2\phi}$. Taking this rescaling into account, we have

$$\mathcal{V}_{\text{Ein}} = \frac{1}{s^2} \mathcal{V}_{\text{String}}. \quad (2.6)$$

This relation will be crucial for our work later on. (We will always work in gravitational mass units, with $M_G = \frac{1}{\sqrt{8\pi G_N}} = 2.4 \times 10^{18} \text{ GeV}$).

Keeping only the main moduli fields $\{S, T_I, U_I\}$, their kinetic terms are determined in terms of the Kähler potential K [24, 25]:

$$K = -\log (S + \bar{S}) - \sum_I \log (T_I + \bar{T}_I) - \sum_I \log (U_I + \bar{U}_I) \quad (2.7)$$

with $K_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$. The classical superpotential depends on the way supersymmetry is broken. Generically string backgrounds with spontaneously broken supersymmetry are flat at the classical level due to the no-scale structure of the effective supergravity theory [25]. Once the thermal and/or quantum corrections are taken into account, we obtain in some cases interesting cosmological solutions.

2.1 Heterotic supersymmetric backgrounds at finite temperature

In order to fix our notations and conventions, we first consider the case of an exact supersymmetric background at finite temperature [21] – [23]. For definiteness we choose the heterotic string with maximal space-time supersymmetry ($N_4 = 4$). All nine spatial directions as well as the Euclidean time are compactified on a ten dimensional torus. At zero temperature, the Euclidean string partition function is zero due to space-time supersymmetry. At finite temperature however the result is a well defined finite quantity. Indeed, at genus one the string partition function is given by:

$$Z = \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau} \frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^4 \frac{\Gamma_{(10,26)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}, \quad (2.8)$$

where $\Gamma_{(10,26)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$ is a shifted Narain lattice (which we specify more precisely below). The non-vanishing of the partition function is due to the non-trivial coupling of the lattice to the spin structures (a, b) . Here, the argument a is zero for space-time bosons and one for space-time fermions. The spin/statistics connection and modular invariance require that the unshifted $\Gamma_{(1,1)}$ sub-lattice of the Euclidean time cycle

$$\Gamma_{(1,1)} \equiv \sum_{m,n} R_0(\text{Im}\tau)^{-\frac{1}{2}} e^{-\pi R_0^2 \frac{|m+n\tau|^2}{\text{Im}\tau}} \quad (2.9)$$

be replaced as follows:

$$\Gamma_{(1,1)} \longrightarrow \sum_{m,n} R_0(\text{Im}\tau)^{-\frac{1}{2}} e^{-\pi R_0^2 \frac{|m+n\tau|^2}{\text{Im}\tau}} e^{i\pi(ma+nb+mn)}. \quad (2.10)$$

Redefining

$$m \rightarrow 2m + g, \quad n \rightarrow 2n + h, \quad (2.11)$$

where g, h are integers defined modulo 2, and introducing the notation $\Gamma_{(1,1)} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ for a shifted lattice,

$$\Gamma_{(1,1)} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] = \sum_{m,n} R_0 (\text{Im}\tau)^{-\frac{1}{2}} e^{-\pi R_0^2 \frac{|2m+g+(2n+h)\tau|^2}{\text{Im}\tau}}, \quad (2.12)$$

the thermal partition function takes the form:

$$Z = \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau} \frac{1}{2} \sum_{(a,b),(h,g)} (-)^{ga+hb+hg} (-)^{a+b+ab} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^4 \frac{\Gamma_{(9,25)} \Gamma_{(1,1)} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}. \quad (2.13)$$

Defining $\hat{a} = a - h$ and $\hat{b} = b - g$ and using the Jacobi identity

$$\frac{1}{2} \sum_{(\hat{a}, \hat{b})} (-)^{\hat{a}+\hat{b}+\hat{a}\hat{b}} \theta \left[\begin{smallmatrix} \hat{a}+h \\ \hat{b}+g \end{smallmatrix} \right]^4 = -\theta \left[\begin{smallmatrix} 1+h \\ 1+g \end{smallmatrix} \right]^4, \quad (2.14)$$

we obtain

$$Z = \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau} \sum_{(h,g)} - (-)^{g+h} \theta \left[\begin{smallmatrix} 1+h \\ 1+g \end{smallmatrix} \right]^4 \Gamma_{(1,1)} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] \frac{\Gamma_{(9,25)}}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}. \quad (2.15)$$

The temperature in string frame is given by $T_{\text{String}} = 1/(2\pi R_0)$.

Since our aim is the study of induced cosmological solutions in 3+1 dimensions, we consider the case for which the radii of three spatial directions are very large: $R_x = R_y = R_z \equiv R \gg 1$. In this case the three dimensional spatial volume factorizes

$$\Gamma_{(3,3)} \cong R^3 (\text{Im}\tau)^{-\frac{3}{2}} = \frac{V_3}{(2\pi)^3} (\text{Im}\tau)^{-\frac{3}{2}}. \quad (2.16)$$

Using the expression for the $\Gamma_{(1,1)} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ shifted lattice we obtain:

$$\begin{aligned} Z &= - (2\pi R_0) V_3 \mathcal{F}_{\text{String}} = V_4 P_{\text{String}} \\ &= - \frac{V_4}{(2\pi)^4} \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau^3} \sum_{(n,m),(h,g)} (-)^{g+h} e^{-\pi R_0^2 \frac{|2m+g+(2n+h)\tau|^2}{\text{Im}\tau}} \theta \left[\begin{smallmatrix} 1+h \\ 1+g \end{smallmatrix} \right]^4 \frac{\Gamma_{(6,22)}}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}, \end{aligned} \quad (2.17)$$

where $V_4 = (2\pi R_0) V_3$ is the four dimensional space-time volume, $\mathcal{F}_{\text{String}}$ the free energy density and P_{String} the pressure in string frame.

Before we proceed further, we make some comments:

- The sector $(h, g) = (0, 0)$ gives zero contribution. This is due to the fact that we started with a supersymmetric background.

- In the odd winding sector, $h = 1$, the partition function diverges when R_0 is between the Hagedorn radius $R_H = (\sqrt{2} + 1)/2$ and its dual $1/R_H$: $\frac{1}{R_H} < R_0 < R_H$. The divergence is due to a winding state that is tachyonic when R_0 takes values in this range, and it signals a phase transition around the Hagedorn temperature [21] – [23]. In this paper we study the regime $R_0 > R_H$, where there is no tachyon and the odd winding sector is exponentially suppressed. The high temperature regime and the cosmological consequences of the phase transition will be examined in future work [32].

- When $R_0 \gg 1$, the contributions of the oscillator states are also exponentially suppressed, provided that the moduli parameterizing the internal $\Gamma_{(6,22)}$ lattice are of order unity.

2.2 The effective field theory in the large R_0 limit

As we already mentioned, the $h = 1$ sector of the theory gives exponentially suppressed contributions of order $\mathcal{O}(e^{-R_0^2})$. Also, the $(h, g) = (0, 0)$ sector vanishes due to supersymmetry. Thus for large R_0 , only the sector $(h, g) = (0, 1)$ contributes significantly. Using the identity:

$$\Gamma_{(1,1)}(R_0) = \Gamma_{(1,1)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \Gamma_{(1,1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \Gamma_{(1,1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Gamma_{(1,1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (2.18)$$

and neglecting the $h = 1$ sectors, we may replace

$$\Gamma_{(1,1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \Gamma_{(1,1)}(R_0) - \Gamma_{(1,1)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \Gamma_{(1,1)}(R_0) - \frac{1}{2}\Gamma_{(1,1)}(2R_0) \quad (2.19)$$

in the integral expression for Z . For each lattice term we decompose the contribution in modular orbits: $(m, n) = (0, 0)$ and $(m, n) \neq (0, 0)$. For $(m, n) \neq (0, 0)$, the integration over the fundamental domain is equivalent with the integration over the whole strip but with $n = 0$. The $(0, 0)$ contribution is integrated over the fundamental domain. Now the $(0, 0)$ contribution of $\Gamma_{(1,1)}(R_0)$ cancels the one of $\frac{1}{2}\Gamma_{(1,1)}(2R_0)$, and we are left with the integration over the whole strip:

$$Z = \frac{V_4}{(2\pi)^4} \int_{||} \frac{d\tau d\bar{\tau}}{4\text{Im}\tau^3} \sum_m e^{-\pi R_0^2 \frac{(2m+1)^2}{\text{Im}\tau}} \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^4 \frac{\Gamma_{(6,22)}}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}. \quad (2.20)$$

The integral over τ_1 imposes the left-right level matching condition. The left-moving part contains the ratio

$$\frac{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^4}{\eta^{12}} = 2^4 + \mathcal{O}(e^{-\pi\tau_2}), \quad (2.21)$$

which implies that the lowest contribution is at the massless level. Thus after the integration over τ_1 ($\tau_2 \equiv t$), the partition function takes the form

$$Z = \frac{V_4}{(2\pi)^4} \int_0^\infty \frac{dt}{2t^3} \sum_m e^{-\pi R_0^2 \frac{(2m+1)^2}{t}} \left(2^4 D_0 + \sum D(\mu) e^{-\pi t \mu^2} \right), \quad (2.22)$$

where $D(\mu)$ denotes the multiplicity of the mass level μ and $2^4 D_0$ is the multiplicity of the massless level. Changing the integration variable by setting $t = \pi R_0^2 (2m+1)^2 x$, we have:

$$Z = \frac{V_4}{\pi^2 (2\pi R_0)^4} \sum_m \frac{1}{(2m+1)^4} \int_0^\infty \frac{dx}{2x^3} e^{-\frac{1}{x}} \left(2^4 D_0 + \sum D(\mu) e^{-x \pi^2 (2m+1)^2 \mu^2 R_0^2} \right). \quad (2.23)$$

Now the second term in the parenthesis is exponentially suppressed when the masses μ are of order (or close) to the string oscillator mass scale. This will be the case when all of the internal radii and the Wilson-line moduli of the $\Gamma_{(6,22)}$ lattice are of order unity. For this specific case, the partition function simplifies to

$$Z = 2^3 D_0 \frac{V_4}{\pi^2 (2\pi R_0)^4} \sum_m \frac{1}{(2m+1)^4} = \frac{2^3 D_0 \pi^2}{48} \frac{V_4}{(2\pi R_0)^4} = \frac{1}{3} \frac{n^* \pi^2}{16} V_4 T_{\text{String}}^4, \quad (2.24)$$

where $n^* = 2^3 D_0$ is the number of the massless boson/fermion pairs in the theory. The free energy density and pressure in string frame are given by

$$P_{\text{String}} = -\mathcal{F}_{\text{String}} = \frac{1}{3} \frac{n^* \pi^2}{16} \frac{T_{\text{String}}^4}{s^2}. \quad (2.25)$$

In the Einstein frame, energy densities are rescaled by a factor $1/s^2$ as in Eq. (2.6). Thus the pressure and free energy density in this frame are given by

$$P_{\text{Ein}} = -\mathcal{F}_{\text{Ein}} = \frac{1}{3} \frac{n^* \pi^2}{16} \frac{T_{\text{String}}^4}{s^2} = \frac{1}{3} \frac{n^* \pi^2}{16} \frac{T^4}{s^2}, \quad (2.26)$$

where $T = T_{\text{String}}/\sqrt{s}$ is the proper temperature in the Einstein frame. This result is expected from the effective field theory point of view. When only massless states are thermally excited, the field theory expression for the pressure is given by

$$P = \frac{1}{3} \left(n_B + \frac{7}{8} n_F \right) \frac{\pi^2 T^4}{30}, \quad (2.27)$$

where n_B and n_F are the numbers of massless bosonic and fermionic degrees of freedom respectively. When $n_B = n_F = n^*$, as in a supersymmetric theory, we recover Eq. (2.26).

2.3 Spontaneous breaking of supersymmetry at zero temperature

In this case we consider the same class of heterotic models, but now the breaking of supersymmetry arises due to the coupling of the space-time fermion number to the momentum and winding quantum numbers of an internal spatial cycle [18] – [23]. Since the temperature is taken to be zero, the spin structures (a, b) do not couple to the quantum numbers of the Euclidean time cycle which will be taken to be very large. We also consider the case where three additional spatial directions are large. Following similar steps to the purely thermal case, the partition function is given by

$$Z = -\frac{V_5}{(2\pi)^5} \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau^{\frac{7}{2}}} \sum_{(n,m),(h,g)} (-)^{g+h} e^{-\pi R_5^2 \frac{|2m+g+(2n+h)\tau|^2}{\text{Im}\tau}} \theta \left[\begin{smallmatrix} 1+h \\ 1+g \end{smallmatrix} \right]^4 \frac{\Gamma_{(5,21)}}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}, \quad (2.28)$$

where now $V_5 = V_4(2\pi R_5)$ is a five dimensional volume and the $\Gamma_{(5,21)}$ lattice parameterizes the internal space. Here also, the $h = 1$ sectors give exponentially suppressed contributions $\mathcal{O}(e^{-R_5^2})$, and the $(h, g) = (0, 0)$ sector vanishes due to supersymmetry. The rest of the steps can be repeated as in the derivation above to find

$$Z = \frac{V_5}{(2\pi)^5} \int_0^\infty \frac{dt}{2t^{\frac{7}{2}}} \sum_m e^{-\pi R_5^2 \frac{(2m+1)^2}{t}} \left(2^4 D_0 + \sum D(\mu) e^{-\pi t \mu^2} \right), \quad (2.29)$$

which after the change of variables $t = \pi R_5^2 (2m+1)^2 x$ gives

$$Z = \frac{V_5}{\pi^{\frac{5}{2}} (2\pi R_5)^5} \sum_m \frac{1}{|2m+1|^5} \int_0^\infty \frac{dx}{2x^{\frac{7}{2}}} e^{-\frac{1}{x}} \left(2^4 D_0 + \sum D(\mu) e^{-x\pi^2 (2m+1)^2 \mu^2 R_5^2} \right). \quad (2.30)$$

For μ of order unity, this simplifies to

$$Z = 2 \left(1 - 2^{-5} \right) \frac{\zeta(5) \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} n^* \frac{V_4}{(2\pi R_5)^4} \quad (2.31)$$

with $n^* = 2^3 D_0$.

This result was expected from the effective field theory point of view. Indeed in a theory with spontaneously broken $N = 4$ supersymmetry, the one loop effective potential receives a non-zero contribution proportional to the mass super-trace $\text{Str}\mathcal{M}^4$, which in turn is proportional to the fourth power of the gravitino mass. The super-traces $\text{Str}\mathcal{M}^n$ vanish for $n < N = 4$. In the example of supersymmetry breaking we examined above, the masses of the states are shifted according to their spin. For initially massless states, the mass after supersymmetry

breaking becomes :

$$M_Q^2 \rightarrow \frac{Q_F^2}{R_5^2}. \quad (2.32)$$

This shows that the string frame gravitino mass is of order $M_{\text{String}} \sim 1/R_5$ and thus $\text{Str}\mathcal{M}^4 \sim c/R_5^4$. Including the contributions from all Kaluza-Klein states, one obtains the result given in formula (2.31). We obtain for the string frame effective potential:

$$\mathcal{V}_{\text{String}} = -\frac{Z}{V_4} = -2(1 - 2^{-5}) \frac{\zeta(5)\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} n^* \frac{1}{(2\pi R_5)^4}. \quad (2.33)$$

In the Einstein frame, we have $\mathcal{V}_{\text{Ein}} = \frac{1}{s^2} \mathcal{V}_{\text{String}}$ – see Eq. (2.6) – so that

$$\mathcal{V}_{\text{Ein}} = -2(1 - 2^{-5}) \frac{\zeta(5)\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} n^* \frac{1}{s^2(2\pi R_5)^4} = -C_V \frac{1}{(s t_1 u_1)^2} = -C_V M^4, \quad (2.34)$$

where $t_1 = \text{Re}(T_1)$, $u_1 = \text{Re}(U_1)$, and $M = 1/(s t_1 u_1)^{1/2}$ is the gravitino mass scale in the Einstein frame.

We stress here that the one loop effective potential depends only on the gravitino mass scale, which in turn depends only on the product of the s , t_1 and u_1 moduli. This suggests to freeze all moduli and keep only the diagonal combination

$$3 \log z = \log s + \log t_1 + \log u_1. \quad (2.35)$$

The Kähler potential of the diagonal modulus Z , (with $z = \text{Re}(Z)$), takes the well known $SU(1, 1)$ structure [25]

$$K = -3 \log(Z + \bar{Z}). \quad (2.36)$$

This gives rise to the kinetic term and gravitino mass scale,

$$-g^{\mu\nu} 3 \frac{\partial_\mu Z \partial_\nu \bar{Z}}{(Z + \bar{Z})^2}, \quad M^2 = 8e^K = \frac{8}{(Z + \bar{Z})^3}. \quad (2.37)$$

Freezing $\text{Im}Z$ and defining the field Φ by

$$e^{2\alpha\Phi} = M^2 = \frac{8}{(Z + \bar{Z})^3}, \quad (2.38)$$

one finds the kinetic term

$$-g^{\mu\nu} 3 \frac{\partial_\mu Z \partial_\nu \bar{Z}}{(Z + \bar{Z})^2} = -g^{\mu\nu} \frac{\alpha^2}{3} \partial_\mu \Phi \partial_\nu \Phi. \quad (2.39)$$

The choice $\alpha^2 = 3/2$ normalizes canonically the kinetic term of the modulus Φ . The potential for this particular model is:

$$\mathcal{V}_{\text{Ein}}(\Phi) = -C_V M^4 = -C_V e^{4\alpha\Phi}, \quad \alpha = \sqrt{\frac{3}{2}}. \quad (2.40)$$

Observe that in this simple model the sign of the potential is negative. As we now explain, we can construct models with a positive potential, but with the rest of the dependence on the modulus Φ being the same. All we have to do is to couple the momentum and winding numbers of the Scherk-Schwarz cycle not only to the space-time fermion number but also to another internal charge. For example consider the $E_8 \times E'_8$ heterotic string on T^6 and instead of coupling just to Q_F , we couple to $Q_F + Q_{E_8} + Q'_{E_8}$, where Q_{E_8} denotes the charge of an E_8 representation decomposed in terms of $SO(16)$ ones, and similarly for Q'_{E_8} . These charges take half integer values for the spinorial representations and integer values for the others. The initial Susy-breaking co-cycle gets modified as follows

$$(-)^{ag+bh+hg} \longrightarrow (-)^{(a+\bar{\gamma}+\bar{\gamma}')g+(b+\bar{\delta}+\bar{\delta}')h+hg}, \quad (2.41)$$

where as before the argument a is one for space-time fermions and zero for space-time bosons, and $(\bar{\gamma}, \bar{\gamma}') = (1, 1)$ for the spinorial representations of $SO(16) \times SO(16)'$ and $(0, 0)$ for the adjoint representations. This operation breaks explicitly the $E_8 \times E'_8$ gauge group to $SO(16) \times SO(16)'$. Proceeding in similar way as in the previous example, one finds:

$$Z = 2 (1 - 2^{-5}) \frac{\zeta(5)\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} \tilde{n}^* \frac{V_4}{(2\pi R_5)^4}, \quad (2.42)$$

where

$$\tilde{n}^* = 2^3 [[2]_{X_{2,3}} + [6]_{T^6} + [120 - 128]_{E_8} + [120 - 128]_{E'_8}] = -2^3 \times 8 = -64. \quad (2.43)$$

In the previous example only positive signs appear in the above formula since there is no coupling of the Scherk-Schwarz lattice quantum numbers to the $E_8 \times E'_8$ charges, giving the value $n^* = 2^3 \times 504$. The reversing of sign for some representations indicates that it is for the bosons that the masses are shifted and not for the fermions in the corresponding multiplet.

We note that in the $N = 4$ case, we cannot change the left-multiplicity since all of the left-moving R-charges are equivalent as required by symmetry. This however is not true for the $N = 2$ and $N = 1$ cases. Consider for instance the class of $N = 2$ supersymmetric

backgrounds obtained by compactifying the heterotic string on a T^4/\mathbb{Z}_2 orbifold (e.g. the \mathbb{Z}_2 -orbifold limit of the K_3 CY-compactification). In this class of models (see for instance [27]) four internal supercoordinates are twisted and the corresponding four internal R-charges are half-shifted. The Euclidean partition function is given by

$$Z = \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau} \frac{1}{4} \sum_{(a,b),(H,G)} (-)^{a+b+ab} \frac{\theta \begin{bmatrix} a \\ b \end{bmatrix}^2 \theta \begin{bmatrix} a+H \\ b+G \end{bmatrix}^2}{\eta(\tau)^4} \\ \times \frac{\Gamma_{(1,1)}(R_0) \Gamma_{(3,3)}(\text{space})}{\eta(\tau)^2 \bar{\eta}(\bar{\tau})^2} Z_{(2,2+n_0)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} Z_{(4,4+n_t)} \begin{bmatrix} H \\ G \end{bmatrix}. \quad (2.44)$$

Here $Z_{(2,2+n_0)}$ is the contribution of two internal coordinates¹ (X_5, X_6) and n_0 -right moving world-sheet bosons ϕ_i . Before supersymmetry breaking, the corresponding $(2, 2+n_0)$ -lattice is unshifted. $Z_{(4,4+n_t)}$ stands for the contribution of four internal coordinates (X_7, X_8, X_9, X_{10}) all of which are \mathbb{Z}_2 -twisted by (H, G) , and n_t -right moving world-sheet bosons ϕ_I which can be \mathbb{Z}_2 -twisted breaking part of the initial gauge group. The θ -function terms come from the contribution of the left-moving world-sheet fermions. Four of them are \mathbb{Z}_2 -twisted by (H, G) . The contribution associated to the space-time bosons is when $a = 0$, while the one associated to the space-time fermions is when $a = 1$.

From the above supersymmetric $N = 2$ partition function, the thermal partition function is obtained in a way similar to the $N = 4$ example, by the following replacement of the Euclidean time sub-lattice:

$$\Gamma_{(1,1)}(R_0) \longrightarrow \Gamma_{(1,1)} \begin{bmatrix} h_1 \\ g_1 \end{bmatrix} (R_0) (-)^{g_1 a + h_1 b + h_1 g_1}. \quad (2.45)$$

In the case of Scherk-Schwarz spontaneous supersymmetry breaking, the partition function can be obtained by a similar replacement of the internal X_5 coordinate lattice, either by utilizing the same operator Q_F

$$\Gamma_{(1,1)}(R_5) \longrightarrow \Gamma_{(1,1)} \begin{bmatrix} h_2 \\ g_2 \end{bmatrix} (R_5) (-)^{g_2 a + h_2 b + h_2 g_2} \quad (2.46)$$

or by utilizing an R-symmetry operator associated to one of the twisted complex planes

$$\Gamma_{(1,1)}(R_5) \longrightarrow \Gamma_{(1,1)} \begin{bmatrix} h_2 \\ g_2 \end{bmatrix} (R_5) (-)^{g_2(a+H) + h_2(b+G) + h_2 g_2}. \quad (2.47)$$

These are in fact the only two possibilities involving left-moving R charges since all others are equivalent choices. However, many other choices exist by utilizing parity-like operators

¹In our notations, the space-time coordinates are $X_{0,\dots,3}$, while the internal ones are $X_{5,\dots,10}$.

involving the right moving gauge charges $\sum \bar{\gamma}_i$, as in the explicit example of $SO(16) \times SO(16)'$ spinorial representations we gave above:

$$\Gamma_{(1,1)}(R_5) \longrightarrow \Gamma_{(1,1)} \left[\begin{smallmatrix} h_2 \\ g_2 \end{smallmatrix} \right] (R_5) (-)^{g_2(a+H+\sum \bar{\gamma}_i)+h_2(b+G+\sum \bar{\delta}_i)+h_2 g_2}. \quad (2.48)$$

In the next section we examine representative examples in the case where thermal and spontaneous Susy breaking operations are present simultaneously.

3 Thermal and spontaneous breaking of Susy

The most interesting situation for cosmological applications is the case where spontaneous supersymmetry breaking and thermal corrections are taken into account simultaneously.

3.1 Untwisted sector

The untwisted sector of the $N = 2$ case, $(H, G) = (0, 0)$ in Eq. (2.44),² has an $N = 4$ structure and thus all choices for the left R-symmetry operators are equivalent. The quantum numbers of the Euclidean time cycle and the internal X_5 -cycle are coupled to the spin structures (a, b) in the same way. After performing the Jacobi theta-function identity the partition function becomes:

$$Z_{\text{untwist}} = -\frac{1}{2} \frac{V_5}{(2\pi)^5} \oint_F \frac{d\tau d\bar{\tau}}{4\text{Im}\tau^{\frac{7}{2}}} \sum_{(n_1, m_1), (h_1, g_1)} \sum_{(n_2, m_2), (h_2, g_2)} (-)^{g_1+g_2+h_1+h_2} \\ e^{-\pi R_0^2 \frac{|2m_1+g_1+(2n_1+h_1)\tau|^2}{\text{Im}\tau}} e^{-\pi R_5^2 \frac{|2m_2+g_2+(2n_2+h_2)\tau|^2}{\text{Im}\tau}} \theta \left[\begin{smallmatrix} 1+h_1+h_2 \\ 1+g_1+g_2 \end{smallmatrix} \right]^4 \frac{\Gamma(5, 21)}{\eta(\tau)^{12} \bar{\eta}(\bar{\tau})^{24}}. \quad (3.1)$$

The factor of $1/2$ is due to the \mathbb{Z}_2 orbifolding of the $N = 4$ theory.

Proceeding as in the simpler examples before and neglecting the $h_1 = 1$ and $h_2 = 1$ sectors for large R_0, R_5 , the non-zero contributions to the partition function occur when $g_1 + g_2 = 1$. Assuming also that all other moduli are of order unity, the only non-exponentially suppressed contributions come from the zero mass left- and right-levels. We obtain

$$Z_{\text{untwist}} = 2^3 D_0 \frac{V_5}{(2\pi)^5} \sum_{g_1, g_2} \frac{(1 - (-)^{g_1+g_2})}{2} \int_0^\infty \frac{dt}{2t^{\frac{7}{2}}} \sum_{m_1, m_2} e^{-\pi R_0^2 \frac{(2m_1+g_1)^2}{t}} e^{-\pi R_5^2 \frac{(2m_2+g_2)^2}{t}}, \quad (3.2)$$

²For $h_1 = h_2 = 0$ (even windings), the sector $(H, G) = (0, 1)$ gives zero net contribution due to the identity $\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} (-)^{ag_1} (-)^{ag_2} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^2 \theta \left[\begin{smallmatrix} a \\ b+1 \end{smallmatrix} \right] \theta \left[\begin{smallmatrix} a \\ b-1 \end{smallmatrix} \right] = 0$.

which after the change of variables $t = \pi (R_0^2(2m_1 + g_1)^2 + R_5^2(2m_2 + g_2)^2)$ x gives

$$Z_{\text{untwist}} = \frac{4D_0 \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \frac{V_5}{(2\pi)^5} \sum_{m_1, m_2} \frac{1}{(R_0^2(2m_1 + 1)^2 + R_5^2(2m_2)^2)^{\frac{5}{2}}} \\ + \frac{4D_0 \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \frac{V_5}{(2\pi)^5} \sum_{m_1, m_2} \frac{1}{(R_0^2(2m_1)^2 + R_5^2(2m_2 + 1)^2)^{\frac{5}{2}}}. \quad (3.3)$$

This expression is symmetric under the $R_0 \leftrightarrow R_5$ exchange. This is suggestive of a temperature/gravitino mass, T/M , duality. This duality will be broken when the supersymmetry breaking arises due to the coupling to a different Q_R charge than Q_F .

To obtain the effective four dimensional pressure, we must factorize out the space-time volume V_4 . To this extent it is convenient to define the complex structure-like ratio

$$u = \frac{R_0}{R_5} = \frac{M}{T}, \quad (3.4)$$

and re-write the partition function in the following way

$$Z_{\text{untwist}} = \frac{4D_0 \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \frac{V_4}{(2\pi R_0)^4} \sum_{m_1, m_2} \frac{u^4}{|(2m_1 + 1)iu + 2m_2|^5} \\ + \frac{4D_0 \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \frac{V_4}{(2\pi R_5)^4} \sum_{m_1, m_2} \frac{1}{|2m_1iu + (2m_2 + 1)|^5}. \quad (3.5)$$

Define the function

$$f(u) \equiv \sum_{m_1, m_2} \frac{u^4}{|(2m_1 + 1)iu + 2m_2|^5}, \quad (3.6)$$

which we can express in terms of Eisenstein functions of order $5/2$:

$$f(u) = u^{\frac{3}{2}} \left(\frac{1}{2^{\frac{5}{2}}} E_{5/2} \left(\frac{iu}{2} \right) - \frac{1}{2^5} E_{5/2}(iu) \right), \quad (3.7)$$

where

$$E_k(U) = \sum_{(m,n) \neq (0,0)} \left(\frac{\text{Im } U}{|m + nU|^2} \right)^k. \quad (3.8)$$

Then the pressure in the Einstein frame can be written as

$$P_{\text{untwist}} = C_T^{\text{unt}} T^4 f(u) + C_V^{\text{unt}} M^4 \frac{f(1/u)}{u}, \quad (3.9)$$

where

$$C_T^{\text{unt}} = C_V^{\text{unt}} = n_{\text{unt}}^* \frac{\Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}}. \quad (3.10)$$

Here $n_{\text{unt}}^* = 4D_0$ is the number of massless boson/fermion pairs in the untwisted sector. It is smaller by a factor of $1/2$ from the corresponding number in the $N = 4$ case due to the \mathbb{Z}_2 -orbifolding. In this particular model the coefficients C_T^{unt} and C_V^{unt} are equal due to the underlying gravitino mass/temperature duality. For fixed u the first term stands for the thermal contribution to the pressure while the second term stands for minus the effective potential.

We note that the coefficient C_T^{unt} is fixed and positive as it is determined by the number of all massless boson/fermion pairs in the untwisted sector of the initially supersymmetric theory: $n_{\text{unt}}^* = 4D_0$. In general, the coefficient C_V^{unt} will depend on the precise way supersymmetry is broken. As we have demonstrated in the previous section, it can take both negative and positive values, depending on how the Susy-breaking operator couples to the right movers: $C_V^{\text{unt}} \sim \tilde{n}_{\text{unt}}^*$. Thus in general the temperature/gravitino mass scale duality will be broken.

Let us discuss the large u limit, which can be obtained by taking R_5 to be small (but still parametrically larger as compared to the string scale), while taking R_0 to be much larger. In this limit we expect to find a four dimensional system at finite temperature, for which only massless bosonic degrees of freedom are thermally excited. *All* fermions attain a mass from the Scherk-Schwarz boundary conditions along the X_5 cycle, and this mass is much bigger than the temperature for large u . Therefore they can be integrated out giving a temperature independent contribution to the pressure of order $1/R_5^4$. Setting $\tilde{u} = 1/u$, we have in the limit $\tilde{u} \rightarrow 0$ ($u \rightarrow \infty$):

$$f(u) = f(1/\tilde{u}) \rightarrow \sum_{m_1} \int_{-\infty}^{\infty} \frac{dx}{((2m_1 + 1)^2 + 4x^2)^{\frac{5}{2}}} = \frac{2}{3} \sum_{m_1} \frac{1}{(2m_1 + 1)^4} = \frac{1}{3} \times \frac{\pi^4}{24}, \quad (3.11)$$

and

$$\begin{aligned} \frac{f(1/u)}{u} &\rightarrow \sum_{m_1} \frac{1}{|2m_1 + 1|^5} + \frac{1}{2^5 u^4} \sum_{m_2 \neq 0} \int_{-\infty}^{\infty} \frac{dx}{(m_2^2 + x^2)^{\frac{5}{2}}} \\ &= 2(1 - 2^{-5}) \zeta(5) + \frac{1}{24 u^4} \sum_{m_2 \neq 0} \frac{1}{m_2^4} \\ &= 2(1 - 2^{-5}) \zeta(5) + \frac{1}{12 u^4} \times \frac{\pi^4}{90}. \end{aligned} \quad (3.12)$$

Using these results, we find

$$P_{\text{untwist}} = \frac{1}{3} \frac{2^3 D_0}{2} \frac{\pi^2 T^4}{30} + 2(1 - 2^{-5}) \frac{\zeta(5) \Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} (4D_0) M^4 \quad (3.13)$$

for the first two leading terms for small $\tilde{u} = 1/u$. We have used the relation $\Gamma(\frac{5}{2}) = 3\sqrt{\pi}/4$. The first contribution arises from the thermally excited massless bosons. As compared to Eq. (2.27) with $n_B = 2^3 D_0$ and $n_F = 0$, it is off by a factor of $1/2$ due to the \mathbb{Z}_2 -orbifolding. Similarly, the second term is off by a factor of $1/2$ as compared to Eq. (2.31) due to the orbifolding. For large u , the potential term is dominant. For generic values of u both fermions and bosons contribute to the thermal piece as in Eq. (3.9), with the contribution depending on the number of massless states at zero temperature and before the breaking of supersymmetry. Finally for small u , the system is essentially a five dimensional purely thermal system.

3.2 Twisted sector, $H = 1$

The contributions of the twisted sectors in the large R_0, R_5 limit depend on the number of the massless twisted states before the supersymmetry breaking, and can be determined in a similar way as before. However, there is a class of models where the \mathbb{Z}_2 -orbifolding acts freely, without any fixed points, and therefore there are no massless states in the twisted sectors. For this class of models, the whole contribution to the one-loop partition function, in the large R_0, R_5 limit, is that of the massless untwisted sector states we have already determined. One example with this property is when the \mathbb{Z}_2 -twists (H, G) are accompanied with a shift of the $\Gamma_{(1,1)}(R_6)$ sub-lattice. This operation leads to the modification of $Z_{(2,10)}$ in Eq. (2.44), where we also set $(n_0, n_t) = (8, 8)$, as follows:

$$Z_{(2,10)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow Z_{(2,10)} \begin{bmatrix} H \\ G \end{bmatrix} = \frac{\Gamma_{(1,1)}(R_5) \Gamma_{(1,1)}(R_6) \begin{bmatrix} H \\ G \end{bmatrix}}{\eta(\tau)^2 \bar{\eta}(\bar{\tau})^2} \frac{1}{2} \sum_{\gamma, \delta} \frac{\bar{\theta} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}^8}{\bar{\eta}(\bar{\tau})^8}. \quad (3.14)$$

If R_6 is sufficiently large, the coupling of the (H, G) -shift of the lattice to the twisted partition function $Z_{(4,12)} \begin{bmatrix} H \\ G \end{bmatrix}$ ensures the absence of massless states in the twisted sector.

In other situations, there are massless states in the twisted sector. Before the supersymmetry breaking, the number of massless bosons is equal to the number of massless fermions with a multiplicity n_{twist}^* . Proceeding as in the untwisted sector, and neglecting the $h_1, h_2 = 1$ sectors, one finds that there is only a non zero contribution when $g_1 + g_2 = 1$. The relative sign of the thermal part $\sim T^4$ and the supersymmetry breaking part $\sim M^4$ depends on the choice of the operators Q_F and Q_R . When there is no coupling to the right-moving gauge

charges we obtain:

$$P_{\text{twist}} = C_T^{\text{twist}} T^4 f(u) + C_V^{\text{twist}} M^4 \frac{f(1/u)}{u}, \quad (3.15)$$

where

$$\begin{aligned} C_T^{\text{twist}} &= n_{\text{twist}}^* \frac{\Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \\ C_V^{\text{twist}} &= \epsilon n_{\text{twist}}^* \frac{\Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \\ \epsilon &= (-)^{(Q_R - Q_F)}. \end{aligned} \quad (3.16)$$

Here, we have for the coefficient ϵ :

- $\epsilon = 1$ when $Q_R = Q_F$.
- $\epsilon = -1$ when $Q_R \neq Q_F$.

In the later case, $(-)^{(Q_R - Q_F)} = (-)^H = -1$, see Eq. (2.47), and $H = 1$ in the twisted sector. The change of sign indicates that it is the bosons that are becoming massive because of the supersymmetry breaking. This is related to a mechanism for moduli stabilization induced by geometrical fluxes [17].

Adding the contributions of the untwisted and twisted sectors together we obtain for the pressure

$$P = C_T T^4 f(u) + C_V M^4 \frac{f(1/u)}{u}, \quad (3.17)$$

where $C_T = C_T^{\text{unt}} + C_T^{\text{twist}}$ and likewise for C_V . The sign of the thermal contribution is always positive,

$$C_T = \frac{n_T^* \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}}, \quad (3.18)$$

$n_T^* = n_{\text{unt}}^* + n_{\text{twist}}^*$. The coefficient multiplying the supersymmetry breaking part is given by

$$C_V = \frac{n_V^* \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \quad (3.19)$$

with $n_V^* = n_{\text{unt}}^* + \epsilon n_{\text{twist}}^*$. In general n_V^* can be positive or negative depending on the model.

3.3 An explicit example

As an example we consider the $E_8 \times E_8$ heterotic string on a T^4/\mathbb{Z}_2 orbifold, whose initially supersymmetric partition function is obtained by setting

$$Z_{(2,10)} = \frac{\Gamma_{(1,1)}(R_5) \Gamma_{(1,1)}(R_6)}{\eta(\tau)^2 \bar{\eta}(\bar{\tau})^2} \frac{1}{2} \sum_{\gamma, \delta} \frac{\bar{\theta}[\gamma]_\delta^8}{\bar{\eta}(\bar{\tau})^8}, \quad (3.20)$$

and

$$Z_{(4,12)} \left[\begin{smallmatrix} H \\ G \end{smallmatrix} \right] = \frac{\Gamma_{(4,4)} \left[\begin{smallmatrix} H \\ G \end{smallmatrix} \right]}{\eta(\tau)^4 \bar{\eta}(\bar{\tau})^4} \frac{1}{2} \sum_{\gamma', \delta'} \frac{\bar{\theta} \left[\begin{smallmatrix} \gamma' \\ \delta' \end{smallmatrix} \right]^6 \bar{\theta} \left[\begin{smallmatrix} \gamma'+H \\ \delta'+G \end{smallmatrix} \right] \bar{\theta} \left[\begin{smallmatrix} \gamma'-H \\ \delta'-G \end{smallmatrix} \right]}{\bar{\eta}(\bar{\tau})^8} \quad (3.21)$$

in Eq. (2.44). We shall use an R-symmetry operator associated to one of the twisted complex planes for breaking the supersymmetry, replacing the $\Gamma_{(1,1)}(R_5)$ lattice as in Eq. (2.47). In the twisted sectors, $(H, G) \neq (0, 0)$, the internal $\Gamma_{(4,4)}$ shifted lattice is given by

$$\Gamma_{(4,4)} \left[\begin{smallmatrix} H \\ G \end{smallmatrix} \right] = \frac{2^4 \eta(\tau)^6 \bar{\eta}(\bar{\tau})^6}{\theta \left[\begin{smallmatrix} 1+H \\ 1+G \end{smallmatrix} \right]^2 \bar{\theta} \left[\begin{smallmatrix} 1+H \\ 1+G \end{smallmatrix} \right]^2}. \quad (3.22)$$

The orbifolding breaks the $E_8 \times E_8$ gauge group to $E_8 \times E_7 \times SU(2)$. Under $E_8 \rightarrow E_7 \times SU(2)$, the 248-dimensional adjoint representation of E_8 decomposes as

$$(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{133}, \mathbf{1}). \quad (3.23)$$

The untwisted sector contains $2^3 \times 504$ massless states giving the value $n_{\text{unt}}^* = 4 \times 504$ for the total number of boson/fermion pairs. These numbers arise as follows. In terms of world-sheet left/right movers the number of bosonic degrees of freedom is given by

$$n_{\text{unt}}^* = 4 \times 504 = [4]_{\psi^{2,3,5,6}} \times ([4]_{X^{2,3,5,6}} + [248]_{E_8} + [133]_{E_7} + [3]_{SU(2)}) + [4]_{\psi^{7,8,9,10}} \times ([4]_{X^{7,8,9,10}} + [2]_{SU(2)} \times [56]_{E_7}). \quad (3.24)$$

The first line gives the bosonic content of a $d = 6$ supergravity multiplet, a tensor multiplet and a vector multiplet in the adjoint of the $E_8 \times E_7 \times SU(2)$ gauge group. The second line gives the bosonic content of four uncharged and one charged hyper-multiplets. The number of fermionic degrees of freedom follows by supersymmetry. At finite temperature and when Susy is broken, the contribution of the massless untwisted states is determined as before (see Eq. (3.5)).

Next we analyze the contribution of states in the twisted sectors, $H = 1$. For large R_0, R_5 , we may neglect the $h_1 = 1$ and $h_2 = 1$ sectors. Also for R_6 of order unity we may set $\Gamma_{(1,1)}(R_6) \cong 1$. Setting $H = 1$, the partition function becomes in this limit

$$Z_{\text{twisted}} = \frac{V_5}{(2\pi)^5} \int_{||} \frac{d\tau d\bar{\tau}}{4 \text{Im}\tau^{\frac{7}{2}}} \frac{1}{\eta(\tau)^6 \bar{\eta}(\bar{\tau})^{18}} \sum_{(m_1, g_1), (m_2, g_2)} e^{-\pi R_0^2 \frac{(2m_1 + g_1)^2}{\text{Im}\tau}} e^{-\pi R_5^2 \frac{(2m_2 + g_2)^2}{\text{Im}\tau}} \\ \times \frac{1}{4} \sum_{(a, b, G)} (-)^{a+b+ab} (-)^{ag_1} (-)^{(a+1)g_2} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^2 \theta \left[\begin{smallmatrix} a+1 \\ b+G \end{smallmatrix} \right]^2$$

$$\times \frac{2^4}{\theta \left[\begin{smallmatrix} 0 \\ 1+G \end{smallmatrix} \right]^2 \bar{\theta} \left[\begin{smallmatrix} 0 \\ 1+G \end{smallmatrix} \right]^2} \frac{1}{2} \sum_{\gamma, \delta} \bar{\theta} [\gamma]_8^8 \frac{1}{2} \sum_{\gamma', \delta'} \bar{\theta} [\gamma']_6^6 \bar{\theta} \left[\begin{smallmatrix} \gamma'+1 \\ \delta'+G \end{smallmatrix} \right] \bar{\theta} \left[\begin{smallmatrix} \gamma'-1 \\ \delta'-G \end{smallmatrix} \right]. \quad (3.25)$$

The non-vanishing contributions arise when $g_1 + g_2 = 1$. Non-exponentially suppressed contributions arise only at the zero mass level. To obtain them, we expand the integrand in powers of $q = e^{2\pi i \tau}$.

When $g_1 + g_2 = 1$, we have for the left movers

$$\begin{aligned} & \frac{1}{\eta(\tau)^6 \theta \left[\begin{smallmatrix} 0 \\ 1+G \end{smallmatrix} \right]^2} \frac{1}{2} \sum_{(a,b)} (-)^{a+b+ab} (-)^{ag_1} (-)^{(a+1)g_2} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^2 \theta \left[\begin{smallmatrix} a+1 \\ b+G \end{smallmatrix} \right]^2 \\ &= \frac{1}{\eta(\tau)^6 \theta \left[\begin{smallmatrix} 0 \\ 1+G \end{smallmatrix} \right]^2} \frac{1}{2} \sum_{(a,b)} (-)^{g_2} (-)^{b+ab} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^2 \theta \left[\begin{smallmatrix} a+1 \\ b+G \end{smallmatrix} \right]^2 \\ &= \frac{(-)^{g_2} \theta_2^2 \theta_3^2}{\eta^6 \theta_4^2} \text{ for } G=0 \quad \text{or} \quad \frac{(-)^{g_2} \theta_2^2 \theta_4^2}{\eta^6 \theta_3^2} \text{ for } G=1 \\ &= 4(-)^{g_2} (1 + \mathcal{O}(q^{1/2})). \end{aligned} \quad (3.26)$$

For the right movers we have

$$\begin{aligned} & \frac{1}{\bar{\eta}(\bar{\tau})^{18} \bar{\theta} \left[\begin{smallmatrix} 0 \\ 1+G \end{smallmatrix} \right]^2} \frac{1}{2} \sum_{\gamma, \delta} \bar{\theta} [\gamma]_8^8 \frac{1}{2} \sum_{\gamma', \delta'} \bar{\theta} [\gamma']_6^6 \bar{\theta} \left[\begin{smallmatrix} \gamma'+1 \\ \delta'+G \end{smallmatrix} \right] \bar{\theta} \left[\begin{smallmatrix} \gamma'-1 \\ \delta'-G \end{smallmatrix} \right] \\ &= \frac{1}{\bar{q}^{3/4}} (1 + (-)^G 4 \bar{q}^{1/2} + \dots) (1 + 240 \bar{q} + \dots) [\bar{q}^{1/4} ((-)^G 2 + 56 \bar{q}^{1/2} + \dots)] \\ &= \frac{1}{\bar{q}^{1/2}} [(-)^G 2 + 8 \bar{q}^{1/2} + 56 \bar{q}^{1/2} + \mathcal{O}(\bar{q})]. \end{aligned} \quad (3.27)$$

When we add the contributions of the $(H, G) = (1, 0)$ and $(H, G) = (1, 1)$ twisted sectors together, we find that the lowest right-mass level is at zero mass as it is the case for the lowest left-mass level.

Using these results we obtain the contribution to the partition function of the massless twisted states:

$$\begin{aligned} Z_{\text{twist}} &= 2^6(56+8) \frac{V_5}{(2\pi)^5} \sum_{g_1, g_2} \frac{(-)^{g_2} - (-)^{g_1}}{2} \int_0^\infty \frac{dt}{2t^{\frac{7}{2}}} \sum_{(m_1, m_2)} e^{-\pi R_0^2 \frac{(2m_1+g_1)^2}{\text{Im}\tau}} e^{-\pi R_5^2 \frac{(2m_2+g_2)^2}{\text{Im}\tau}} \\ &= \frac{2^5(56+8) \Gamma\left(\frac{5}{2}\right)}{\pi^{\frac{5}{2}}} \left(\frac{V_4}{(2\pi R_0)^4} f(u) - \frac{V_4}{(2\pi R_5)^4} \frac{f(1/u)}{u} \right). \end{aligned} \quad (3.28)$$

The contribution to the pressure is as in Eqs (3.15), (3.16) with $n_{\text{twist}}^* = 2^5(56+8)$, the number of massless boson/fermion pairs in the twisted sector, and $\epsilon = -1$. The massless

bosonic content consists of 32 scalars in the $(56, 1)$ representation of $E_7 \times SU(2)$ and 128 scalars in the representation $(1, 2)$. The number of massless fermionic degrees of freedom follows by supersymmetry.

Adding the contributions of the untwisted and twisted sectors together, we obtain

$$\begin{aligned} n_T^* &= 4 \times 504 + 16 \times 128 = 4064 \\ n_V^* &= 4 \times 504 - 16 \times 128 = -32. \end{aligned} \tag{3.29}$$

In addition, we have the choice with $\epsilon = 1$ in Eq. (3.16), giving $n_T^* = n_V^* = 4064$. We can also change n_V^* by considering Susy-breaking operators involving the right-moving gauge charges as in Eq. (2.48).

3.4 Small mass scales from Wilson line deformations

A generic supersymmetric heterotic background may contain in its spectrum massive supermultiplets whose mass is obtained by switching on non-trivial continuous Wilson-lines [27] – [29]. This is a stringy realization of the Higgs mechanism, breaking the initial gauge group G to a smaller one spontaneously. This statement is not absolutely correct for discrete Wilson lines corresponding to extended symmetry points where the gauge symmetry may enhance or even get modified.

For our purposes, we restrict to arbitrary and small Wilson line deformations starting from a given supersymmetric background where R_I , $I = 6, 7, \dots, 10$ are of the order the string scale. This restriction ensures that the contributions to the thermal partition function of the momentum and winding states in these five internal directions will be exponentially suppressed in the limit where R_0 and R_5 are large.

A systematic study of the effects of Wilson lines can be found in [27] – [29]. In the zero winding sector, a Wilson line just modifies the Kaluza-Klein momenta, and the corresponding Kaluza-Klein mass becomes

$$\frac{m_I^2}{R_I^2} \longrightarrow \frac{(m_I + y_I^a Q_a)^2}{R_I^2}, \tag{3.30}$$

where Q_a is the charge operator associated to the Wilson-line y_I^a . We distinguish three different situations according to the direction I :

- i) $I = 5$, where R_5 is larger than the string scale.
- ii) $I = 6$, with R_6 of the order of the string scale (with $N = 4$ and $N = 2$ initial supersymmetry).
- iii) $I = 6, \dots, 10$, with R_I of the order of the string scale (with $N = 4$ initial susy only).

In the first case, $I = 5$, after a Poisson re-summation, the net modification to the partition function is obtained by the following replacement in Eq. (3.5):

$$\int_0^\infty \frac{dt}{t^{\frac{7}{2}}} \sum_{m_1, m_2} e^{-\pi R_0^2 \frac{(2m_1+g_1)^2}{t} - \pi R_5^2 \frac{(2m_2+g_2)^2}{t}} \longrightarrow \int_0^\infty \frac{dt}{t^{\frac{7}{2}}} \sum_{m_1, m_2} e^{-\pi R_0^2 \frac{(2m_1+g_1)^2}{t} - \pi R_5^2 \frac{(2m_2+g_2)^2}{t}} \left[e^{2i\pi(2m_2+g_2) y_5^a Q_a} \right]. \quad (3.31)$$

The term in the brackets can be replaced with

$$\cos(2\pi(2m_2 + g_2) y_5^a Q_a) = 1 - 2 \sin^2(\pi(2m_2 + g_2) y_5^a Q_a). \quad (3.32)$$

In the ii) and iii) cases, we can set the momentum and winding numbers to zero, $m_I = n_I = 0$, so that the extra modification in the partition function is the insertion of the term:

$$\left[e^{-\pi t \sum_I \left(\frac{y_I^a Q_a}{R_I} \right)^2} \right] \simeq \left[1 - \pi t \sum_I \left(\frac{y_I^a Q_a}{R_I} \right)^2 \right]. \quad (3.33)$$

Incorporating the effects of the Wilson lines up to quadratic order, we get for the overall pressure:

$$\begin{aligned} P = & T^4 \frac{\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} \sum_{m_1, m_2} \frac{u^4 (n_T^* - 2 \sum_s \sin^2(2\pi m_2 y_5^a Q_a^s))}{|(2m_1 + 1)iu + 2m_2|^5} \\ & - T^2 \frac{\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}} \sum_{m_1, m_2} \frac{u^2 (M_T^2 - 2 \sum_s M_s^2 \sin^2(2\pi m_2 y_5^a Q_a^s))}{|(2m_1 + 1)iu + 2m_2|^3} \\ & + M^4 \frac{\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} \sum_{m_1, m_2} \frac{n_V^* - 2 \sum_s \text{sign}(s) \sin^2((2m_2 + 1)\pi y_5^a Q_a^s)}{|2m_1 iu + (2m_2 + 1)|^5} \\ & - M^2 \frac{\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}} \sum_{m_1, m_2} \frac{M_V^{(2)} - 2 \sum_s \text{sign}(s) M_s^2 \sin^2((2m_2 + 1)\pi y_5^a Q_a^s)}{|2m_1 iu + (2m_2 + 1)|^3}. \end{aligned} \quad (3.34)$$

In this expression, we have defined

$$M_s^2 = \frac{1}{4\pi} \sum_I \frac{(y_I^a Q_a^s)^2}{R_I^2} \quad (3.35)$$

for the pair of boson/fermion states s and also introduced

$$M_T^2 = \sum_s M_s^2, \quad M_V^{(2)} = \sum_s \text{sign}(s) M_s^2, \quad (3.36)$$

where $\text{sign}(s)$ indicates whether the state s contributes positively or negatively to n_V^* and $M_V^{(2)}$, both being possibly negative.

The following comments are in order:

- In the above expression, $y_5^a Q_a^s$ summarizes the contribution of the R_5 -Wilson line in the term corresponding to the pair of boson/fermion states s . It does not introduce a new scale.

- The M_s 's introduce new mass scales in the theory, qualitatively different than T and M . The masses M_s are supersymmetric mass scales rather than susy-breaking scales like T and M . One must keep in mind that after minimization with respect to y_I^a , physical scales like M_s are always proportional to an (infrared) renormalization group invariant scale like, for instance, Λ_{QCD} or the transmutation scale Q_0 , appearing in the electroweak gauge symmetry radiative breaking (of supersymmetric theories in the presence of soft breaking terms) [25] [26].

- The first two terms (which arise from the $(g_1, g_2) = (1, 0)$ sector) can be identified in the effective field theory as the thermal contribution to the pressure, P_{thermal} . Again the number n_T^* is always positive being the number of the massless boson/fermion pairs in the initially supersymmetric background. This purely thermal piece is always positive.

- The two last terms can be identified as minus the effective potential $-\mathcal{V}_{\text{eff}}$. This is naturally regularized in the infrared by the temperature scale T . This infrared regularization differs from that considered in [30], and used in [6], which is valid at zero temperature.

- The number n_V^* can be either positive or negative depending on the way supersymmetry is broken. This shows that the sign of the one loop effective potential depends on the way supersymmetry is broken, as it can be seen in supergravity by utilizing super-trace arguments [31] [26].

3.5 Scaling properties of the thermal effective potential

The final expression for P contains various mass scales: the two supersymmetry breaking scales which are the temperature T and the gravitino mass scale M , as well as the super-

symmetric masses M_s which are generated by the Wilson-lines in the directions 6, 7, 8, 9, 10. The first identity follows immediately from the definition of P :

$$\left(T \frac{\partial}{\partial T} + M \frac{\partial}{\partial M} + \sum_s M_s \frac{\partial}{\partial M_s} \right) P = 4P \quad (3.37)$$

which can be best seen by writing P as

$$P \equiv T^4 p_4(u) + T^2 p_2(u), \quad u = \frac{M}{T}, \quad (3.38)$$

where

$$p_4 = \frac{\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} \left(F(u, y_5^a) + \tilde{F}(u, y_5^a) \right), \quad p_2 = -\frac{\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}} \left(G(u, y_5^a) + \tilde{G}(u, y_5^a) \right), \quad (3.39)$$

and using the definitions

$$F(u, y_5^a) = \sum_{m_1, m_2, s} \frac{u^4 \cos(4\pi m_2 y_5^a Q_a^s)}{[(2m_1 + 1)^2 u^2 + 4m_2^2]^{5/2}},$$

$$\tilde{F}(u, y_5^a) = \sum_{m_1, m_2, s} \frac{u^4 \text{sign}(s) \cos(2\pi(2m_2 + 1)y_5^a Q_a^s)}{[4m_1^2 u^2 + (2m_2 + 1)^2]^{5/2}}; \quad (3.40)$$

$$G(u, y_5^a) = \sum_{m_1, m_2, s} \frac{u^2 M_s^2 \cos(4\pi m_2 y_5^a Q_a^s)}{[(2m_1 + 1)^2 u^2 + 4m_2^2]^{3/2}},$$

$$\tilde{G}(u, y_5^a) = \sum_{m_1, m_2, s} \frac{u^2 \text{sign}(s) M_s^2 \cos(2\pi(2m_2 + 1)y_5^a Q_a^s)}{[4m_1^2 u^2 + (2m_2 + 1)^2]^{3/2}}. \quad (3.41)$$

Using standard thermodynamics identities, we can obtain the energy density ρ :

$$\rho \equiv T \frac{\partial P}{\partial T} - P = T^4 r_4(u) + T^2 r_2(u) \quad (3.42)$$

where

$$r_4 = 3p_4 - u p_4', \quad r_2 = p_2 - u p_2' \quad (3.43)$$

and the primes stand for derivatives with respect to u . In the sequel, we allow the Susy-breaking scales T and M to vary with time while fixing the supersymmetric masses M_s , and investigate the back-reaction to the initially flat metric and moduli fields.

4 Gravitational equations and critical solution

We assume that the back-reacted space-time metric is homogeneous and isotropic,

$$ds^2 = -dt^2 + a(t)^2 d\Omega_k^2, \quad H = \left(\frac{\dot{a}}{a}\right), \quad (4.1)$$

where Ω_k denotes the three dimensional space with constant curvature k and H is the Hubble parameter.

From the fact that $-P$ plays the role of the effective potential and the relation between the gravitino mass scale M and the no-scale modulus Φ ,

$$M = e^{\alpha\Phi}, \quad \alpha = \sqrt{\frac{3}{2}},$$

we obtain the field equation for Φ :

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{\partial P}{\partial \Phi} = \alpha u \left(\frac{\partial P}{\partial u} \right)_T = -\alpha (T^4(r_4 - 3p_4) + T^2(r_2 - p_2)). \quad (4.2)$$

We have made use of Eq. (3.43).

For other flat moduli φ_i , with Φ independent kinetic terms, the equation of motion is straightforward to solve,

$$\ddot{\varphi}_i + 3H\dot{\varphi}_i = 0 \quad \implies \quad \frac{1}{2}\dot{\varphi}_i^2 = \frac{c_i^2}{a^6}, \quad (4.3)$$

where the c_i 's are integration constants.

Knowing the thermal effective potential $-P$, the energy density ρ as well as the field equation for the modulus Φ , we can derive the (one-loop) corrected space-time metric by solving the gravitational field equations. These are the Friedmann-Hubble equation,

$$3H^2 = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\sum_i \dot{\varphi}_i^2 + \rho - \frac{3k}{a^2}, \quad (4.4)$$

and the equation that follows from varying with respect to the spatial components of the metric:

$$2\dot{H} + 3H^2 = -\frac{k}{a^2} - P - \frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\sum_i \dot{\varphi}_i^2. \quad (4.5)$$

For our purposes, it will be useful to replace Eq. (4.5) by the linear sum of Eqs (4.4) and (4.5), so that the kinetic terms of Φ and φ_i drop out:

$$\dot{H} + 3H^2 = -\frac{2k}{a^2} + \frac{1}{2}(\rho - P). \quad (4.6)$$

4.1 Critical solution

The fundamental ingredients in our analysis are the scaling properties of the thermal effective potential $-P = -T^4 p_4 - T^2 p_2$. These scaling properties suggest to search for a solution where all varying mass scales of the system, $M(\Phi)$, T and $1/a$, remain proportional during time evolution:

$$e^{\alpha\Phi} \equiv M(\Phi) = \frac{1}{\gamma a} \implies H = -\alpha\dot{\Phi}, \quad M(\Phi) = uT, \quad (4.7)$$

with γ and u fixed in time. Our aim is to determine the constants γ and u .

On the trajectory (4.7), the Φ -equation is given by

$$\dot{H} + 3H^2 = \alpha^2 \left((r_4 - 3p_4) \frac{M^4}{u^4} + (r_2 - p_2) \frac{M^2}{u^2} \right) \quad (4.8)$$

and the gravity Eq. (4.6) by

$$\dot{H} + 3H^2 = -2k\gamma^2 M^2 + \frac{1}{2}(r_4 - p_4) \frac{M^4}{u^4} + \frac{1}{2}(r_2 - p_2) \frac{M^2}{u^2}. \quad (4.9)$$

The compatibility of these two equations requires an identification of the coefficients of the monomials in M . The quartic terms give an equation for u , while the quadratic terms determine the sign of the parameter k and the magnitude of $|k\gamma^2|$,

$$r_4 = \frac{6\alpha^2 - 1}{2\alpha^2 - 1} p_4, \quad \left(r_4 = 4p_4, \quad i.e. \quad p_4 + up'_4 = 0, \quad \text{for } \alpha^2 = \frac{3}{2} \right), \quad (4.10)$$

$$-2k\gamma^2 = \frac{2\alpha^2 - 1}{2} (r_2 - p_2) \frac{1}{u^2}, \quad \left(-2k\gamma^2 = \frac{(r_2 - p_2)}{u^2}, \quad \text{for } \alpha^2 = \frac{3}{2} \right). \quad (4.11)$$

Eq. (4.10) reminds us of the equation of state for thermal radiation in five dimensions. In the absence of Wilson lines, where $r_2 = p_2 = 0$, we have that $\rho = 4P$, which is indeed the $5d$ state equation for thermal radiation.

When non trivial Wilson lines are turned on, their equations of motion fix them to be proportional to the renormalization group invariant scale Q_0 . Q_0 is supposed to be much smaller than M , $Q_0 \ll M$, so that the y^4 -terms can be consistently neglected. Within this approximation the relevant terms of the y_I^a -effective potential are quadratic in y_I^a modulo multiplicative terms arising from wave-function renormalization. If we define

$$p_2 = \sum_{I,a} \left(\frac{y_I^a}{R_I} \right)^2 p_{2I}^a(u), \quad (4.12)$$

we obtain, in a collectively qualitative way, the effective potential for each Wilson line (no sum over I and a):

$$-V_I^a = Z_I^a(y_I^a/R_I, Q_0) T^2 \left(\frac{y_I^a}{R_I} \right)^2 p_{2I}^a(u), \quad Z_I^a(y_I^a/R_I, Q_0) = A_I^a \left(1 - \log \frac{(y_I^a/R_I)^2}{c_I^a Q_0^2} \right). \quad (4.13)$$

So we have that

$$(y_I^a/R_I) \frac{\partial}{\partial (y_I^a/R_I)} V_I^a = 0 \quad \Longleftrightarrow \quad \left(\frac{y_I^a}{R_I} \right)^2 = c_I^a Q_0^2, \quad (4.14)$$

fixing the Wilson lines to be proportional to the transmutation scale Q_0 . We stress at this point that the c_I^a are positive dimensionless constants, which, in general, are functions of the dimensionless ratio $u = M/T$. The u dependence of the c_I^a is due to threshold corrections induced in the large scale regime: $M, T \gg Q_0$. As we will show below, the u dependence of the c_I^a plays a crucial role in our analysis.

When all other equations of motion are satisfied, the gravity Eq. (4.6) is equivalent to the total energy-momentum conservation:

$$\frac{d}{dt} \left(\rho + \frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} \sum_i \dot{\varphi}_i^2 \right) + 3H \left(\rho + P + \dot{\Phi}^2 + \sum_i \dot{\varphi}_i^2 \right) = 0. \quad (4.15)$$

This imposes an extra constraint when Wilson lines are switched on. On the critical trajectory (4.7), and by using (4.14), one obtains :

$$r_2^R + p_2^R = 0, \quad (4.16)$$

where we have defined the renormalized quantities

$$p_2^R = \sum_{I,a} Z_I^a \left(\frac{y_I^a}{R_I} \right)^2 p_{2I}^a = \sum_{I,a} A_I^a c_I^a(u) Q_0^2 p_{2I}^a(u), \quad r_2^R = p_2^R - u p_2^{R'}, \quad (4.17)$$

to appear as well in Eqs (4.8), (4.9) and (4.11).

Next, let us consider the Friedmann-Hubble Eq. (4.4) along the critical trajectory (4.7). It becomes

$$\left(\frac{6\alpha^2 - 1}{6\alpha^2} \right) 3H^2 = -\frac{3k}{a^2} + \rho + \frac{1}{2} \sum_i \dot{\varphi}_i^2 = -\frac{3k}{a^2} + T^4 r_4 + T^2 r_2^R + \sum_i \frac{c_i^2}{a^6}. \quad (4.18)$$

The dilatation factor in front of $3H^2$ can be absorbed in the definition of \hat{k} , c_r and c_m , once we take into account Eqs (4.20) , (4.21) and (4.22) below:

$$3H^2 = -\frac{3\hat{k}}{a^2} + \frac{c_r}{a^4} + \frac{c_m}{a^6}, \quad (4.19)$$

where, for $\alpha^2 = 3/2$,

$$\hat{k} = -\frac{3}{2\gamma^2 u^2} r_2^R = \frac{3}{2\gamma^2 u^2} p_2^R, \quad (4.20)$$

$$c_r = \frac{9}{8\gamma^4 u^4} r_4 = \frac{9}{8\gamma^4 u^4} 4p_4, \quad (4.21)$$

and

$$c_m = \frac{9}{8} \sum_i c_i^2. \quad (4.22)$$

Clearly, a necessary condition for the effective curvature \hat{k} not to vanish is to have non trivial Wilson lines, at least in the direction 6, and to ensure Eq. (4.16).

We note that Eq. (4.19) also controls the dynamics of a FRW universe, where space has constant curvature \hat{k} and is formally filled with a thermal bath of radiation (since the sign of c_r can a priori be positive as well as negative). There can be an extra contribution, arising from the kinetic terms of some extra flat moduli, that scales as $1/a^6$.

Finally let us address a seeming puzzle. We said that along the critical trajectory and in the absence of Wilson lines, our thermodynamic quantities satisfy $\rho = 4P$. How can this situation correspond to a $4d$ universe filled with thermal radiation? The answer is that we must take into account the kinetic energy density of the modulus field Φ . When there are no Wilson lines and the kinetic terms of the other moduli are switched off, the Friedmann-Hubble equation gives $\dot{\Phi}^2 = (r_4 T^4)/4$ along the critical trajectory, and so the total energy density and pressure satisfy

$$\begin{aligned} \rho_{\text{tot}} &= r_4 T^4 + \frac{1}{2} \dot{\Phi}^2 = \frac{9r_4}{8} T^4 \\ P_{\text{tot}} &= p_4 T^4 + \frac{1}{2} \dot{\Phi}^2 = \frac{3r_4}{8} T^4. \end{aligned} \quad (4.23)$$

Thus $\rho_{\text{tot}} = 3P_{\text{tot}}$ which is the $4d$ equation of state for thermal radiation.

4.2 Numerical study

4.2.1 Without Wilson lines y_5^a

Let us consider the case where the Wilson lines along the X_5 -direction are switched off. From Eq. (3.40), we obtain

$$p_4(u) = \frac{\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}} \left(n_T^* f(u) + n_V^* \tilde{f}(u) \right), \quad \tilde{f}(u) = u^3 f(1/u), \quad (4.24)$$

thanks to the identities $n_T^* f(u) \equiv F(u, y_5^a = 0)$ and $n_V^* \tilde{f}(u) \equiv \tilde{F}(u, y_5^a = 0)$. Since $f + uf'$ and $\tilde{f} + u\tilde{f}'$ vanish at the origin, Eq. (4.10) admits a universal solution $u = 0$ for arbitrary n_V^* in the range $-n_T^* \leq n_V^* \leq n_T^*$. This solution corresponds to $M(t) \equiv 0$ at some finite T , and is associated to a 5-dimensional purely thermal system ($R_5 = \infty$). The 4d effective description we have considered is not valid in this case.

We are thus looking for non trivial solutions $u > 0$ of Eq. (4.10), which we write in the form

$$\frac{n_V^*}{n_T^*} = -\frac{f + uf'}{\tilde{f} + u\tilde{f}'} . \quad (4.25)$$

One can show that such non-trivial solutions (and consequently, non-trivial cosmological evolutions) exist only for models satisfying

$$-\frac{1}{15} < \frac{n_V^*}{n_T^*} < 0 . \quad (4.26)$$

The non vanishing root of Eq. (4.25) is an increasing function of the ratio n_V^*/n_T^* satisfying $u \rightarrow +\infty$ when $n_V^*/n_T^* \rightarrow 0$ and $u \rightarrow 0$ when $n_V^*/n_T^* \rightarrow -1/15$, (See Fig. 1).

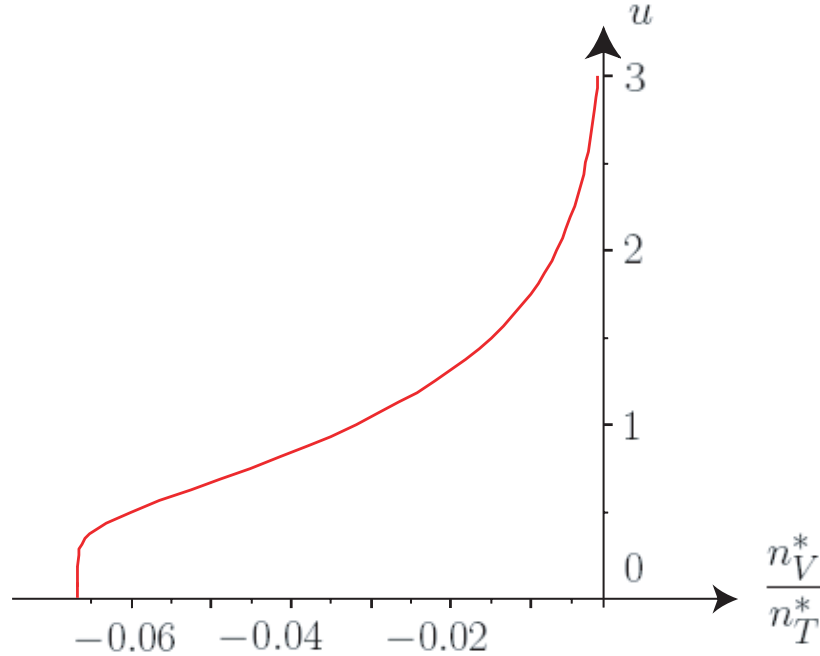


Figure 1: The non trivial root u of Eq. (4.10) as a function of the ratio n_V^*/n_T^* .

The corresponding value of c_r in Eq. (4.21) is finite and always positive. This can be seen

by noting that the quantity $n_T^* f + n_V^* \tilde{f}$ appearing in the expression for c_r equals $-u(n_T^* f' + n_V^* \tilde{f}') > 0$. The positivity follows since f' and \tilde{f}' are always negative and positive respectively.

When Wilson lines in the X_6 -direction are switched on, we can further study the sign of \hat{k} . Defining $G(u, y_5^a = 0) \equiv M_T^2 g(u)$ and $\tilde{G}(u, y_5^a = 0) \equiv M_V^{(2)} \tilde{g}(u)$, the quantity p_2^R in (4.17) becomes

$$p_2^R(u) = -\frac{\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}} \left(M_T^2(u) g(u) + M_V^{(2)}(u) \tilde{g}(u) \right), \quad \tilde{g}(u) = u g(1/u), \quad (4.27)$$

where $M_T^2(u)$, $M_V^{(2)}(u)$ are obtained from M_T^2 and $M_V^{(2)}$ under the replacement $(y_I^a/R_I)^2 \rightarrow A_I^a c_I^a(u) Q_0^2$. Since \hat{k} and p_2^R have same sign (see Eq. (4.20)), one finds that

$$\hat{k} \geq 0 \iff \left\{ u \geq 1 \text{ and } -1 \leq \frac{M_V^{(2)}(u)}{M_T^2(u)} \leq -\frac{g(u)}{\tilde{g}(u)} \right\}. \quad (4.28)$$

Note that a sufficient condition for \hat{k} to be negative is that $u \leq 1$, or $M_V^{(2)} \geq 0$. For the particular value $u = 1$, Eq. (4.25) requires that $n_V^*/n_T^* \simeq -0.0320$.

Let us detail the models we considered in the previous sections by computing the quantities n_V^* , n_T^* , M_T^2 , $M_V^{(2)}$ to determine u and \hat{k} .

The $r_2^R + p_2^R = 0$ constraint (4.16) ensures the existence of non trivial Wilson lines and imposes extra restrictions on the space of critical solutions. Supposing for simplicity that the Wilson lines are functions of u which are all proportional, $c_I^a(u) \equiv c(u)$, one finds on the critical solution with u determined by $r_4 = 4p_4$ (Eq. (4.21):

$$r_2^R + p_2^R = 0 \iff 2 - \frac{d \log(g + s\tilde{g})}{d \log u} = \eta \quad \text{where} \quad s = \frac{M_V^{(2)}}{M_T^2}, \quad \eta = \frac{d \log c}{d \log u}, \quad (4.29)$$

$s \in [-1, 1]$ being a u -independent number in this case. The above relation (4.29) reduces further the permitted values of $s = M_V^{(2)}/M_T^2$. For η small, one can show that (4.29) is satisfied when u is sufficiently large. The analysis for a generic value of η is quite complex, and depends crucially on the particular choice of y_I^a in the configuration space. In the explicit examples we will consider below, we will skip this analysis. However, this is not a real obstruction since, as we will see later, the simultaneous presence of non trivial Wilson lines in the directions 5 and 6 modifies the critical value of u in a highly non trivial way. The study of the η -constraint on the space of (u, y_5^a) goes beyond the scope of this work. Keeping that in mind we disregard the constraint when $y_5^a = 0$ in the examples which are displayed below.

Model 1:

We consider an $N = 4$ heterotic string model, for example the $E_8 \times E_8$ theory on T^6 , at finite temperature and when supersymmetry is spontaneously broken by choosing $Q_R = Q_F$. For this model

$$n_T^* = n_V^* = 2^3 \times 504. \quad (4.30)$$

Since $n_V^* > 0$, there is no critical solution $u > 0$.

Model 2:

In the same $E_8 \times E_8$ heterotic string model, the R -symmetry operator used to break supersymmetry is chosen to be $Q_F + Q_{E_8} + Q'_{E_8}$ in order to have $n_V^* < 0$ (see Eq. (2.43) for the $T = 0$ case):

$$n_T^* = 2^3 \times 504; \quad n_V^* = 2^3 \times (-8). \quad (4.31)$$

Then $n_V^*/n_T^* = -1/63 \simeq -0.0159$, and so the model admits a critical solution $u > 0$. One finds numerically that $u \simeq 1.46$ and $\gamma^4 c_r \simeq 441$.

Let us consider Wilson lines y_I^a in the direction $I = 6$ and define

$$(Y^a)^2 = \frac{1}{4\pi} \left(\frac{y_6^a}{R_6} \right)^2, \quad a = 1, \dots, 16, \quad (4.32)$$

where $a = 1, \dots, 8$ stand for the Cartan generators of the first E_8 factor and $a = 9, \dots, 16$ for the second. The derivation of the charges Q_a of the initially massless states (see the Appendix) gives:

$$M_T^2 = 2^3 \times 60 \left((Y^1)^2 + \dots + (Y^{16})^2 \right), \quad (4.33)$$

and

$$M_V^{(2)} = 2^3 \times (-4) \left((Y^1)^2 + \dots + (Y^{16})^2 \right). \quad (4.34)$$

Their ratio is Wilson line independent, $M_V^{(2)}/M_T^2 = -1/15$, which is larger than $-g(u)/\tilde{g}(u)$, and so the effective curvature is negative.

Model 3:

In the $N = 2$ orbifold model (see section 3.3 for details), we set $Q_R = Q_F + Q_H + Q_{E_7}$ with Q_{E_7} being 1 for the spinorial representations of E_7 , decomposed in terms of $SO(12)$ ones, and 0 for the vectorial ones. We find the following:

$$n_T^* = 4064; \quad n_V^* = 992. \quad (4.35)$$

Since n_V^*/n_T^* is positive, this model doesn't admit a critical point $u > 0$ at this stage.

Model 4:

This is the $N = 2$ orbifold model constructed in Sect. 3.3, with

$$n_T^* = 4064, \quad n_V^* = -32. \quad (4.36)$$

Since $n_V^*/n_T^* = -1/127 \simeq -7.87 \cdot 10^{-3}$, there is again a non trivial critical solution $u > 0$. Numerically, we find $u \simeq 1.90$ and $\gamma^4 c_r \simeq 130$.

Before switching on Wilson lines in the direction $I = 6$, the gauge group is $E_7 \times SU(2) \times E_8$. Let us consider arbitrary Wilson lines, Y^a , $a = 1, \dots, 16$: Y^1, \dots, Y^6 for the Cartan generators of the $SO(12)$ subalgebra of E_7 , Y^7 for the Cartan generator of the $SU(2)$ subalgebra of E_7 , Y^8 for the $SU(2)$ factor and Y^9, \dots, Y^{16} for the E_8 . The mass square scales, computed in the Appendix, are:

$$M_T^2 = 624 \sum_{a=1}^6 (Y^a)^2 + 1248 (Y^7)^2 + 736 (Y^8)^2 + 240 \sum_{a=9}^{16} (Y^a)^2 \quad (4.37)$$

and

$$M_V^{(2)} = -144 \sum_{a=1}^6 (Y^a)^2 - 288 (Y^7)^2 + 224 (Y^8)^2 + 240 \sum_{a=9}^{16} (Y^a)^2. \quad (4.38)$$

The smallest value of $M_V^{(2)}/M_T^2$ is reached when any Y^a , $a = 1, \dots, 7$, is non trivial (while $Y^a = 0$, $a = 8, \dots, 16$). In this case $M_V^{(2)}/M_T^2 = -3/13$, which is larger than $-g(u)/\tilde{g}(u)$, so that \hat{k} is negative.

4.2.2 With Wilson Lines y_5^a

Any model originally characterized by the quantities n_T^* , n_V^* , M_T^2 and $M_V^{(2)}$ can be deformed by switching on the Wilson lines y_5^a , $a = 1, \dots, 16$. We are looking for solutions of Eq. (4.10) written in terms of the functions defined in (3.40):

$$F(u, y_5^a) + uF'(u, y_5^a) + \tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a) = 0. \quad (4.39)$$

We observe that the thermal contribution described by $F(u, y_5^a) + uF'(u, y_5^a)$ vanishes at $u = 0$ and is bounded for large u . In all of the following examples, it is also positive. On the contrary, the “effective potential” corrections $\tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a)$ vanish at $u = 0$

and diverge to $+\infty$ or $-\infty$ at infinity. Thus, in presence of arbitrary Wilson lines y_5^a , the universal solution $u = 0$ remains and we are looking for non trivial ones $u > 0$. These can only arise when $\tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a)$ takes negative values.

Model 1:

Among the Wilson lines y_5^a , ($a = 1, \dots, 8$ for the first E_8 factor and $a = 9, \dots, 16$ for the second), we choose to switch on either

- i)* y_5^1 (with $y_5^2 = \dots = y_5^{16} = 0$) or
- ii)* $y_5^1 = \dots = y_5^8$ (with $y_5^9 = \dots = y_5^{16} = 0$).

In these cases and at fixed Wilson lines, $\tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a)$ increases from 0 to $+\infty$, and so there is still no solution $u > 0$.

Model 2:

For the Wilson lines defined in case *ii)* above, one finds that the critical solution $u > 0$ present before deformation is slightly shifted. However, in case *i)*, the root $u > 0$ is sent to $+\infty$ when y_5^1 approaches the numerical value $\simeq 0.227$ from below. For y_5^1 above this critical bound, there is no non trivial solution anymore.

Model 3:

As said before, this model does not admit a critical solution $u > 0$ when all Wilson lines in the fifth dimension are switched off. We consider four patterns of deformations, by switching on a Wilson line for a single Cartan generator:

- i)* y_5^1 (Cartan generator of $SO(12) \subset E_7$)
- ii)* y_5^7 (Cartan generator of $SU(2) \subset E_7$)
- iii)* y_5^8 (Cartan generator of $SU(2)$)
- iv)* y_5^9 (Cartan generator of E_8)

In cases *i)* and *iv)*, a large enough Wilson line deformation generates a non-trivial solution. We find the critical bound $y_5^1 \simeq 0.895$ for case *i)*, and $y_5^9 \simeq 0.772$ for case *iv)*. In all these cases, the phase transition occurs when the limit at $u \rightarrow \infty$ of the “effective potential” contribution $\tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a)$ is switched from $+\infty$ to $-\infty$. In some sense, a solution $u = +\infty$ appears at the transition, and then decreases for a larger Wilson line, and finally reaches a non-zero minimal value. In cases *ii)* and *iii)*, since $\tilde{F}(u, y_5^a) + u\tilde{F}'(u, y_5^a)$ is positive for any value of the Wilson line, the deformation does not create a non-trivial solution.

Model 4:

We consider the same four patterns of Wilson lines. This time, the critical solution $u > 0$ exists before deforming the model. We find that in the above cases, switching on a large enough Wilson line makes this solution disappear. However, two distinct behaviors are found:

In cases *i*) and *ii*), we find again that the phase transition corresponds to a change in the behavior of the “effective potential” contribution at ∞ . In a similar way as before, the solution $u > 0$ existing before deformation will increase when switching on the Wilson line, then go to $+\infty$ at the transition point, and disappear for a larger Wilson line. We find the critical bound $y_5^1 \simeq 0.105$ for case *i*), and $y_5^7 \simeq 0.074$ for case *ii*).

For *iii*) and *iv*), switching on the Wilson line makes the critical solution $u > 0$ decrease towards zero, so that we are left only with the universal solution when the Wilson line is above some bound. In case *iii*), this maximal value is $y_5^8 \simeq 0.205$, while it is $y_5^9 \simeq 0.207$ for case *iv*).

Some remarks are in order

- In all cases presented above, when a critical solution $u > 0$ exists, the effective radiation term c_r given by Eq. (4.21) is strictly positive.
- In some models, incorporating the Wilson lines in the fifth direction allows the critical solution $u > 0$ to be close to 0 or large. In other words, a hierarchy between the scales M and T can be found by tuning the moduli y_5^a .

5 Cosmological evolutions

When a non degenerate solution $u > 0$ exists, the model admits a well defined low energy description in four dimensions. The dynamics in this regime is controlled by the Friedmann-Hubble equation, whose behavior depends drastically on the signs of c_r and \hat{k} . In all models we considered here, with initial $N = 4$ or $N = 2$ supersymmetry, c_r turns out to be positive. However, this situation may not be generic in more complex stringy examples with initial $N = 1$ supersymmetry. For completeness, we briefly describe all possible cosmologies arising for any positive or negative value of c_r . In addition, we allow non trivial time dependent profiles for the moduli ϕ_i , giving $c_m > 0$.

5.1 Solutions for $c_r > 0$

- For $\hat{k} = 0$, $c_r > 0$, $c_m \geq 0$

When Wilson lines in the directions 6, 7, ..., 10 are absent, the curvature \hat{k} vanishes. In real time, the Friedmann-Hubble Eq. (4.19),

$$3H^2 = \frac{c_r}{a^4} + \frac{c_m}{a^6}, \quad (5.1)$$

can be used to express the time variable t as an integral function of the scale factor a ,

$$t(a) = \sqrt{\frac{3}{c_r}} \int_0^a \frac{v^2 dv}{\sqrt{v^2 + a_0^2}}, \quad (a \geq 0), \quad a_0 = \sqrt{\frac{c_m}{c_r}}. \quad (5.2)$$

In this form, it is straightforward to draw $a(t)$ (see Fig. 2). The explicit solution is

$$t(a) = t_0 \left(\frac{a}{a_0} \sqrt{1 + \frac{a^2}{a_0^2}} - \operatorname{arcsinh} \left(\frac{a}{a_0} \right) \right), \quad t_0 = \frac{\sqrt{3}}{2} \frac{c_m}{c_r^{3/2}}, \quad (5.3)$$

describing an expanding universe that starts with a big bang. As can be seen from Eq. (5.1), the slope \dot{a} is infinite when a vanishes. At large t , the scale factor behaves as in the $c_m = 0$ particular case:

$$a(t) = \left(\frac{c_r}{3} \right)^{1/4} \sqrt{2t}, \quad (t \geq 0). \quad (5.4)$$

Since the transformation $t \rightarrow -t$ is a symmetry of the Friedmann-Hubble equation, the previous expanding solutions have contracting counterparts and thus ending at $t = 0$ with a big crunch. Finally, since the RHS of Eq. (5.1) is positive, there is no solution in Euclidean time.

The effective field theory description always breaks down before the occurrence of a space-like singularity, when the temperature (in string frame) is of order the Hagedorn temperature. At this temperature scale, new stringy dynamics must be taken into consideration which can result into a phase transition, realizing the scenario of [16].

- For $\hat{k} > 0$, $c_r > 0$, $c_m \geq 0$

When we switch on Wilson lines we could have $\hat{k} > 0$. Rewriting the Friedmann-Hubble Eq. (4.19) in the form

$$(a^2 \dot{a})^2 = \hat{k}(a^2 + a_-^2)(a_+^2 - a^2), \quad a_{\pm} = \sqrt{\frac{\sqrt{c_r^2 + 12\hat{k}c_m} \pm c_r}{6\hat{k}}}, \quad (5.5)$$

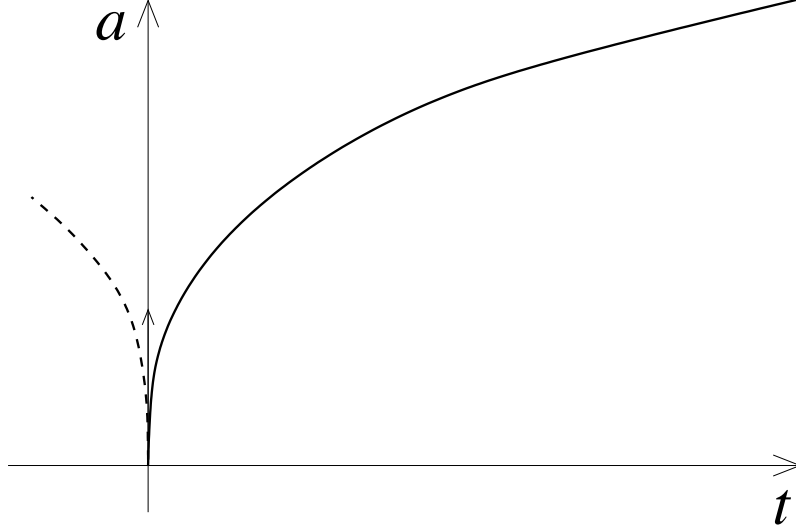


Figure 2: *Cosmological evolution for the case $\hat{k} = 0$, $c_r > 0$, $c_m \geq 0$.*

one expects a cosmological evolution satisfying $0 \leq a \leq a_+$ should exist, while a solution with a scale factor greater than a_+ should make sense in imaginary time only. In real time, one can actually express t as a function of a as follows

$$t(a) = \pm \left(t_i + \frac{1}{\sqrt{\hat{k}}} \int_0^a \frac{v^2 dv}{\sqrt{(v^2 + a_-^2)(a_+^2 - v^2)}} \right), \quad (0 \leq a \leq a_+), \quad (5.6)$$

where

$$t_i = -\frac{1}{\sqrt{\hat{k}}} \int_0^{a_+} \frac{v^2 dv}{\sqrt{(v^2 + a_-^2)(a_+^2 - v^2)}}. \quad (5.7)$$

From these expressions, one can see that the cosmological evolution starts with a big bang at $t = t_i$. It expands until $t = 0$ where the maximum size of the universe a_+ is reached, and then it contracts until a big crunch occurs at $t = -t_i$.

To find a Euclidean solution, one needs to consider a scale factor greater than a_+ . It is then possible to find it by proceeding as before, or use the fact that such a solution can be obtained by analytic continuation of the expression (5.6) at $t = 0$, where $a = a_+$. One can write

$$t = \pm \left(t_i + \frac{1}{\sqrt{\hat{k}}} \int_0^{a_+} \frac{v^2 dv}{\sqrt{(v^2 + a_-^2)(a_+^2 - v^2)}} + \frac{1}{\sqrt{\hat{k}}} \int_{a_+}^{a_E} \frac{v^2 dv}{\sqrt{-(v^2 + a_-^2)(v^2 - a_+^2)}} \right) \equiv -i\tau, \quad (5.8)$$

from which we derive

$$\tau(a_E) = \pm \frac{1}{\sqrt{\hat{k}}} \int_{a_+}^{a_E} \frac{v^2 dv}{\sqrt{(v^2 + a_-^2)(v^2 - a_+^2)}}, \quad (a_E \geq a_+). \quad (5.9)$$

Fig. 3 represents the solutions $a(t)$ and $a_E(\tau)$.

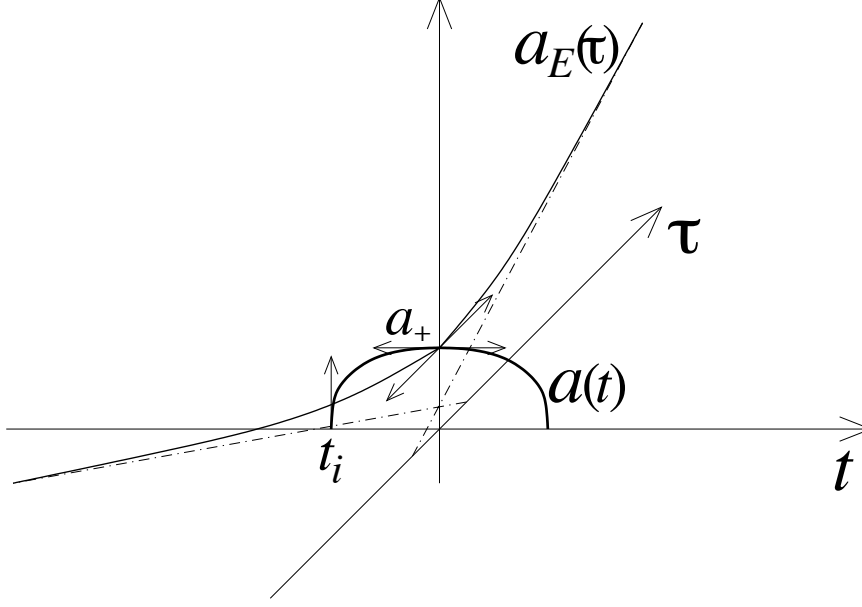


Figure 3: *Cosmological evolution for the case $\hat{k} > 0$, $c_r > 0$, $c_m \geq 0$ (in bold line). A big bang and a big crunch are occurring at $t = t_i$ and $t = -t_i$ respectively. This solution is connected to a Euclidean one at $t = -i\tau = 0$ that is asymptotically linear.*

In the particular case where $c_m = 0$, the solutions (5.6) and (5.9) are taking the explicit forms

$$a(t) = a_+ \sqrt{1 - \left(\frac{t}{t_i}\right)^2}, \quad (t_i \leq t \leq -t_i), \quad a_+ = \sqrt{\frac{c_r}{3\hat{k}}}, \quad t_i = -\frac{1}{\hat{k}} \sqrt{\frac{c_r}{3}}, \quad (5.10)$$

and

$$a_E(\tau) = a_+ \sqrt{1 + \left(\frac{\tau}{t_i}\right)^2}, \quad (5.11)$$

whose shapes are similar to the generic case with $c_m > 0$.

- For $\hat{k} < 0$, $c_r > 0$, $c_m \geq 0$

This case is easier to deal with. Eq. (4.19) can be rewritten as

$$(a^2 \dot{a})^2 = |\hat{k}|(a^2 + a_-^2)(a^2 + a_+^2), \quad a_{\pm} = \sqrt{\frac{c_r \pm \sqrt{c_r^2 - 12|\hat{k}|c_m}}{6|\hat{k}|}}, \quad (5.12)$$

and admits the expanding solution

$$t(a) = \frac{1}{\sqrt{\hat{k}}} \int_0^a \frac{v^2 dv}{\sqrt{(v^2 + a_-^2)(v^2 + a_+^2)}}. \quad (5.13)$$

(See Fig. 4.) After a big bang, the scale factor is growing linearly in time.

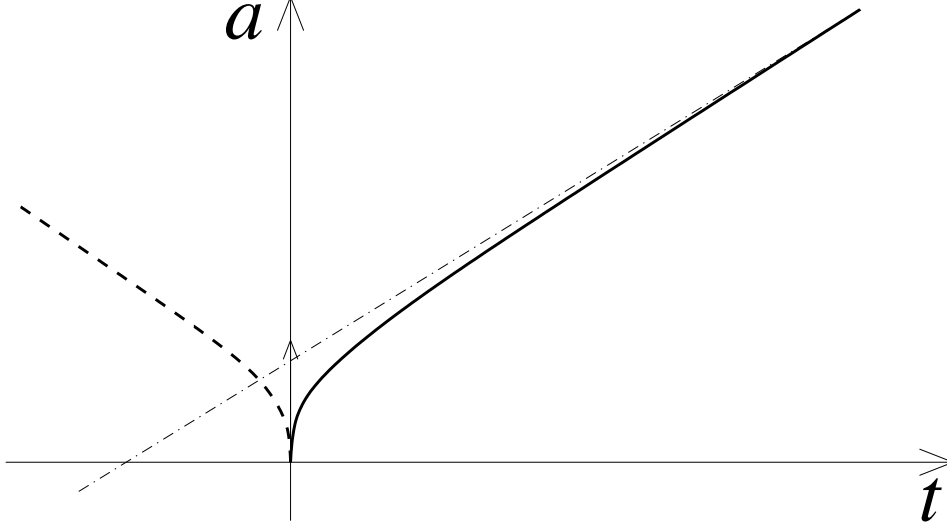


Figure 4: *Cosmological evolution for the case $\hat{k} < 0$, $c_r > 0$, $c_m \geq 0$.*

The result for the particular case $c_m = 0$ can be written more explicitly. The solution takes the form

$$a(t) = a_+ \sqrt{\left(\frac{t+t_0}{t_0}\right)^2 - 1}, \quad (t \geq 0), \quad a_+ = \sqrt{\frac{c_r}{3|\hat{k}|}}, \quad t_0 = \frac{1}{|\hat{k}|} \sqrt{\frac{c_r}{3}}. \quad (5.14)$$

These cosmological solutions do not admit a sensible Euclidean continuation.

5.2 Exotic cosmologies with $c_r < 0$

Although in all explicit models we presented before c_r is positive, it is interesting to analyze the exotic situation with negative c_r , which is not a priori forbidden in more general cases with $N = 1$ initial supersymmetry.

- For $\hat{k} = 0$, $c_r < 0$, $c_m \geq 0$

A cosmological evolution in real time only exists if c_m is switched on. The scale factor

satisfies $0 \leq a \leq a_0$ where

$$a_0 = \sqrt{\frac{c_m}{|c_r|}}. \quad (5.15)$$

Between a big bang at $t_i < 0$ and a big crunch at $-t_i > 0$, a reaches a maximum a_0 at $t = 0$. At this time, an analytic continuation is allowed: A Euclidean solution satisfies $a_E \geq a_0$ and goes to infinity for large positive or negative Euclidean time (see Fig. 5).

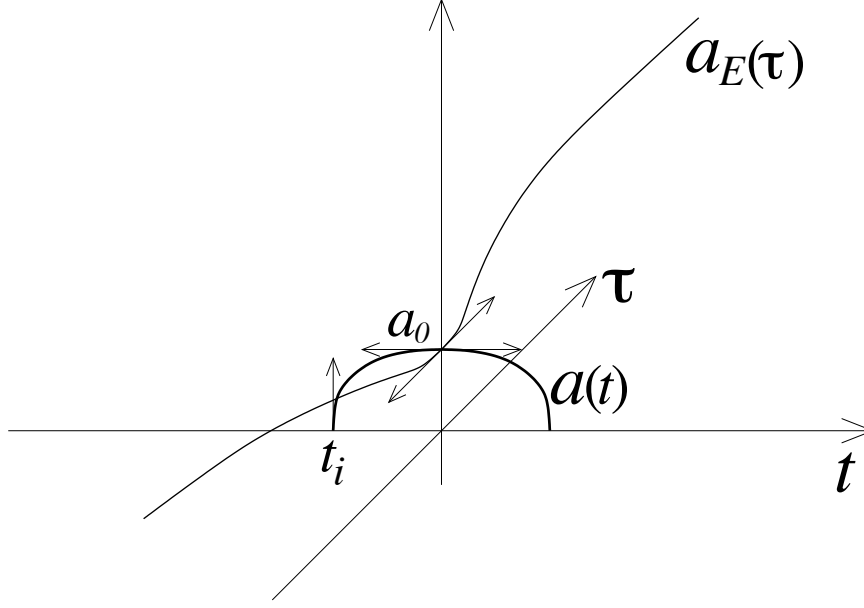


Figure 5: *Cosmological evolution for the case $\hat{k} = 0$, $c_r < 0$, $c_m \geq 0$ (in bold line). A big bang and a big crunch are occurring at $t = t_i$ and $t = -t_i$ respectively. This solution is connected to a Euclidean one at $t = -i\tau = 0$.*

- For $\hat{k} > 0$, $c_r \leq 0$, $c_m \geq 0$

Most of the considerations of this case are identical to the one derived for $\hat{k} > 0$, $c_r > 0$, $c_m \geq 0$. In particular, the Friedmann-Hubble equation is still given by Eq. (5.5) and both the cosmological solution (5.6) and the Euclidean one (5.9) are valid. They are shown in Fig. 6. As long as $c_m > 0$, the only qualitative difference with the case $c_r > 0$ is that the Euclidean solution has two symmetric inflexion points. However, when c_m vanishes, the evolution in real time ceases to exist.

- For $\hat{k} < 0$, $c_r < 0$, $c_m \geq 0$

This case presents the most interesting features and involves either a first or second order phase transition in the early universe. The former case is the only one considered in this

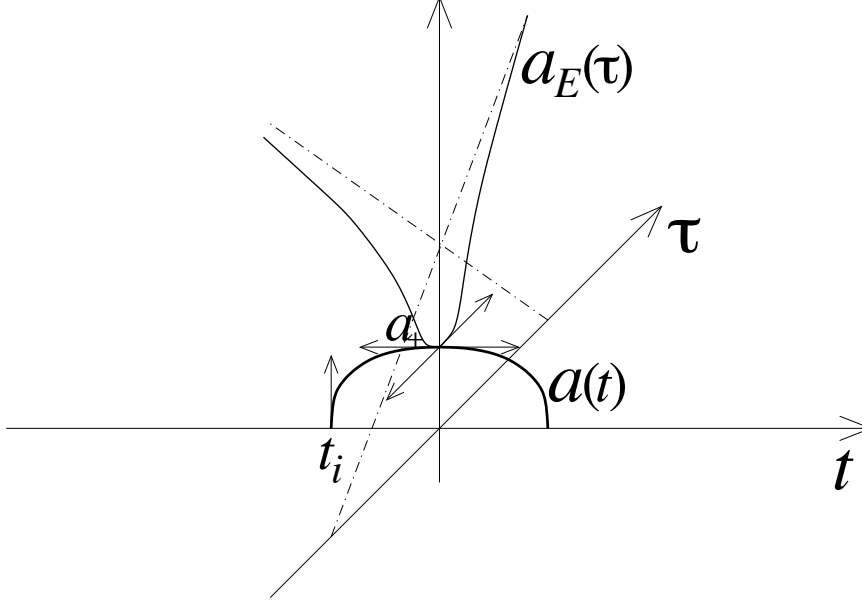


Figure 6: *Cosmological evolution for the case $\hat{k} > 0$, $c_r \leq 0$, $c_m > 0$ (in bold line). A big bang and a big crunch are occurring at $t = t_i$ and $t = -t_i$ respectively. This solution is connected to a Euclidean one at $t = -i\tau = 0$ that is asymptotically linear and has two symmetric inflexion points.*

paper, where a Euclidean solution has a finite action and thus can be interpreted as an instanton involved in a tunneling effect. These behaviors are qualitatively similar to the inflationary case studied in [6, 12]. To be more specific we consider the Friedmann-Hubble equation in the form

$$3(a^2\dot{a}) = 3|\hat{k}|a^4 - |c_r|a^2 + c_m, \quad (5.16)$$

and discuss various regimes, depending on the value of the discriminant of the RHS

$$\delta \equiv c_r^2 - 12|\hat{k}|c_m. \quad (5.17)$$

i) When $\delta > 0$, there are two critical values for the scale factor

$$a_{\pm} = \sqrt{\frac{|c_r| \pm \sqrt{c_r^2 - 12|\hat{k}|c_m}}{6|\hat{k}|}}. \quad (5.18)$$

Eq. (5.16) then admits two distinct cosmological evolutions. The first one satisfies $0 \leq a \leq a_-$ and corresponds to the usual dynamics between a big bang at $t_i < 0$ and a big crunch at $-t_i$. The scale factor reaches a maximum a_- at $t = 0$. The second one describes an asymptotically linear contracting solution followed by an asymptotically linear expanding

one. The two branches are smoothly connected at $t = 0$, where the scale factor reaches a minimum value a_+ . Therefore, this solution is non-singular. (See Fig. 7.)

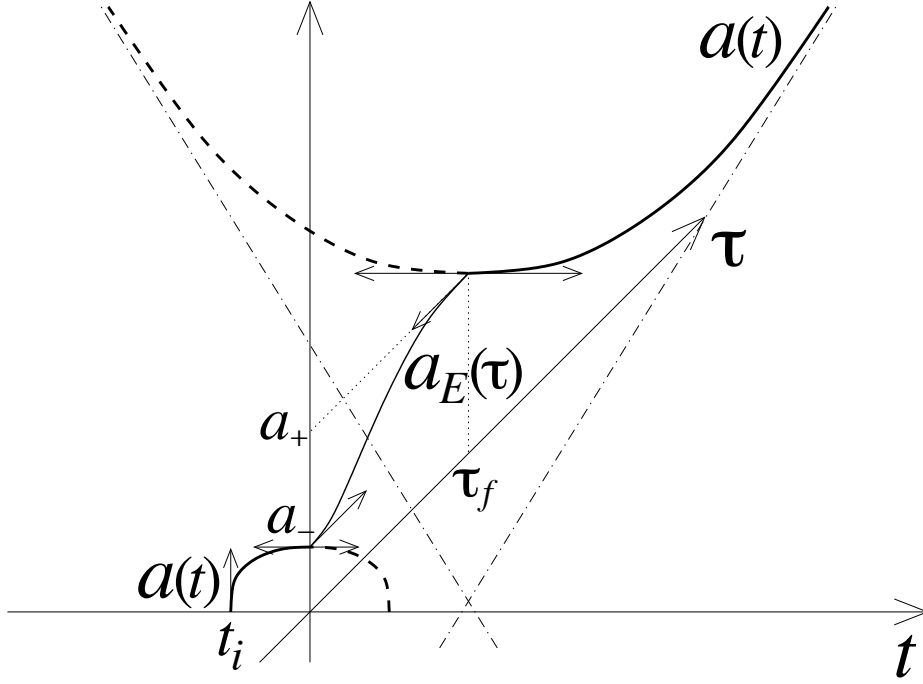


Figure 7: There are two cosmological evolutions (in bold lines) for the case $\hat{k} < 0$, $c_r \leq 0$, $c_m \geq 0$, when $\delta > 0$. The first one starts with a big bang at $t = t_i$ and ends with a big crunch at $t = -t_i$. The second one has a contracting phase followed by an expanding one. These two branches are connected to each other by a first order phase transition via an instanton.

The two cosmological evolutions are also related to one another by a double analytic continuation: $t = -i\tau$ and then $\tau = \tau_f + it$. Between $\tau = 0$ and $\tau = \tau_f$, a Euclidean solution whose action can be shown to be finite is allowed. It is thus an instanton between the two branches in real time and contributes to a first order phase transition. We note that when $c_m = 0$, the big bang / big crunch solution disappears since a_- vanishes.

ii) Let us turn now to the second case where the discriminant (5.17) satisfies $\delta < 0$. Eq. (5.16) does not admit any critical point and the scale factor is never stationary. There is a single cosmological evolution (and no Euclidean solution). It increases from a big bang at $t = 0$, while for large t , its time dependence becomes linear. Thus, close to the big bang, the evolution is similar to the first solution occurring when $\delta > 0$, while for large t its behavior is similar to the second expanding solution. Since there is an inflexion point at $t = t_{\text{inf}}$ when $a = \sqrt{2c_m/|c_r|}$, the cosmological evolution for $\delta < 0$ can be interpreted as a second order

phase transition between the same initial and final states encountered in the first order phase transition for $\delta > 0$. (See Fig. 8.)

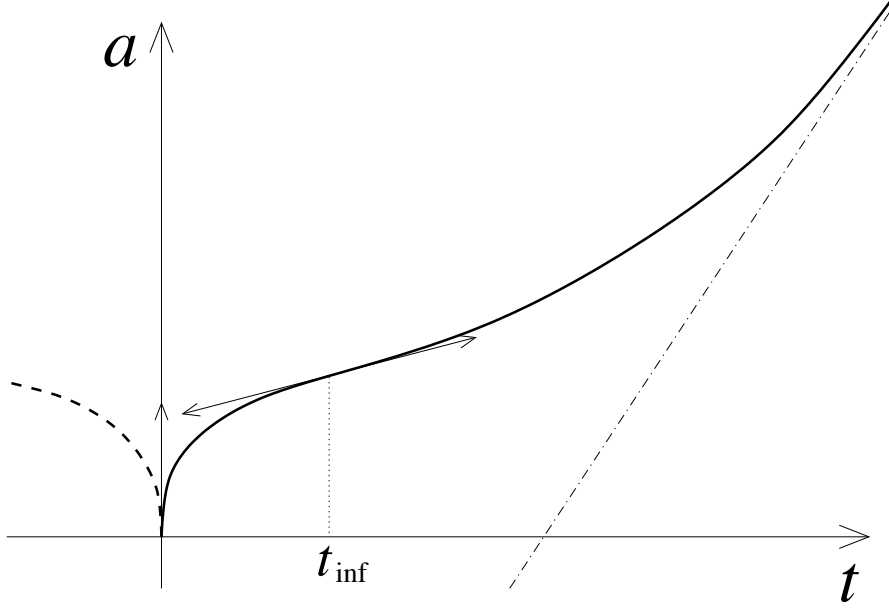


Figure 8: *Cosmological evolution for the case $\hat{k} < 0$, $c_r \leq 0$, $c_m \geq 0$, when $\delta < 0$. It describes a second order phase transition occurring at t_{inf} , between a phase that starts with a big bang to another phase that expands linearly in time.*

iii) In the critical case $\delta = 0$, Eq. (5.16) admits two expanding cosmological evolutions which are asymptotic to a static one, $a \equiv a_0$, where

$$a_0 \equiv a_{\pm} = \sqrt{\frac{|c_r|}{6|\hat{k}|}} = a_{\text{inf}} = \sqrt{\frac{2c_m}{|c_r|}}, \quad (5.19)$$

together with two contracting ones obtained by time reversal. The first expanding solution starts with a big bang, while the second one is linear for large positive time. (See Fig. 9.)

6 Conclusions

We have obtained several cosmological solutions in a large class of four dimensional heterotic string compactifications with spontaneously broken $N = 4$ or $N = 2$ space-time supersymmetry. The cosmological evolution is induced once radiative quantum and thermal corrections are taken into consideration. These corrections are calculated at the perturbative string

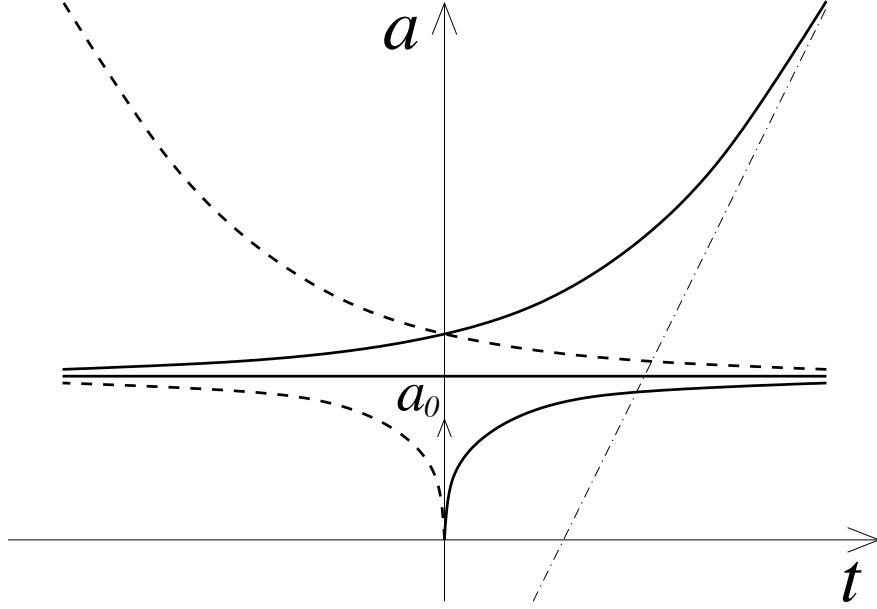


Figure 9: *There are two expanding (contracting) cosmological evolutions for the case $\hat{k} < 0$, $c_r \leq 0$, $c_m \geq 0$, when $\delta = 0$. All are asymptotic to the static solution $a \equiv a_0$.*

level and shown to possess universal scaling properties. The reason is an underlying duality between the temperature and the supersymmetry breaking scale.

Our solutions correspond to homogeneous and isotropic Friedmann-Robertson-Walker universes. They are characterized by the ratio of the supersymmetry breaking scale to the temperature, and this ratio remains constant during time evolution. Even though Kaluza-Klein states associated to the supersymmetry breaking cycle are thermally excited, the equation of state governing cosmological evolution is identical to that of massless thermal radiation in four dimensions. This is due to the special relation between the no-scale modulus field associated to the supersymmetry breaking scale and the Hubble parameter: $\dot{\Phi}^2 = 2H^2/3$. Universes with spherical, toroidal or hyperbolic spatial sections can be found once we incorporate Wilson line deformations.

In this paper we focused on the low temperature phase of the models. When the temperature is close to the Hagedorn temperature our effective field theory analysis breaks down and new stringy dynamics must be taken into consideration. It would be interesting to investigate if phase transitions can occur in these models as the temperature approaches the Hagedorn temperature, and whether such phase transitions result in non-singular time-dependent geometries. To this extent it could prove useful to incorporate in our work the proposal of [23],

where such a phase transition is shown to occur in $N = 4$ heterotic string models, and study the cosmological implications.

It would be interesting to extend our analysis to the $N = 1$ heterotic orbifold models, and for the cases where supersymmetry is broken spontaneously. In this class of models, one expects to find inflationary phases, once radiative and thermal corrections are properly taken into account. The analysis of [6, 12] reveals interesting transitions between such inflationary phases and radiation dominated phases with similar properties to those found in this work. In our examples, the coefficient of the $1/a^4$ term in the Friedmann-Hubble equation is positive. Perhaps among the $N = 1$ examples it is possible to find models characterized by negative values of this coefficient. Then non trivial cosmological phenomena would occur, including first or second order phase transitions that allow for the possibility to realize the proposal for the creation of a universe from “nothing” [11] in string theory [6, 12].

The relation between the supersymmetry breaking scale with the temperature is a key property of our solutions. Suppose that such a scaling property persisted in an early universe epoch, and that initially supersymmetry was broken around the string scale. During such epoch, the Susy-breaking scale gets lower and lower as the universe expands and cools. At lower temperatures new dynamics may become relevant that can stabilize this scale. Such a scenario can give us a new perspective on how to handle the hierarchy and naturalness problems.

Note added: A follow up of the present work can be found in [33]. The radiation era we have described is not only obtained for specific initial boundary conditions that select the critical trajectory. Instead, the radiation era is an attractor of the dynamics. It is reached asymptotically for generic initial boundary conditions.

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Appendix A

In this Appendix, we provide details on the derivation of the charges associated to the Cartan generators of the gauge groups of the heterotic string models we considered. We also derive the contributions to M_T^2 and $M_V^{(2)}$ (see Eq. (3.34)) associated to Wilson lines in the internal direction 6 only. Since the result is linear in the $(y_6^i)^2$'s, the result for arbitrary Wilson lines in the directions 6, 7, 8, 9, 10 is obtained under the replacement $(y_6^i)^2/(4\pi R_6^2) \rightarrow \sum_I (y_I^i)^2/(4\pi R_I^2)$.

Consider the $N = 4$ heterotic model with gauge group $E_8 \times E_8$. When built out from 32 worldsheet fermions as in the standard procedure, the vertex operators for the gauge fields associated to the Cartan generators of the first E_8 factor are given by

$$\lambda^{2i-1} \lambda^{2i}, \quad i = 1, \dots, 8, \quad (\text{A.1})$$

whereas those associated to the Cartan generators of the second E_8 are given by

$$\lambda^{2i-1} \lambda^{2i}, \quad i = 9, \dots, 16. \quad (\text{A.2})$$

In terms of representations of $SO(16)$, the adjoint representation of E_8 decomposes as

$$\mathbf{248} = \mathbf{120} \oplus \mathbf{128}, \quad (\text{A.3})$$

where the **120** is the adjoint representation of $SO(16)$ and the **128** is the spinorial representation with positive chirality. The corresponding vertex operators can be written explicitly by bosonizing the 32 fermions into 16 free bosons H_i , $i = 1, \dots, 16$. This formalism has the advantage to make the roots of the Lie algebra appear in a clear way. For the **120** we obtain

$$\mathbf{120} : \left(e^{i(\pm H_j \pm H_k)}, \quad (j \neq k) \right) \oplus i\partial H_j, \quad j, k \in \{1, \dots, 8\}. \quad (\text{A.4})$$

The 8 latter vertex operators correspond to the Cartan generators. For the **128** we have

$$\mathbf{128} : e^{\frac{i}{2}(\epsilon_1 H_1 + \epsilon_2 H_2 + \dots + \epsilon_8 H_8)}, \quad (\text{A.5})$$

with the GSO constraint $\prod_{i=1}^8 \epsilon_i = 1$. Our goal here is to compute the quantity

$$M_T^2 = \frac{1}{4\pi R_6^2} \sum_{s \in \mathbf{248}} (Q_a^s y_6^a)^2, \quad (\text{A.6})$$

where we denote by Q_a^s the charges of a state s with respect to the Cartan generators. We are going to consider the case of one E_8 gauge group; the generalization to $E_8 \times E_8$ is straightforward.

To begin with, we recall that the Cartan states are neutral. Then we are interested in the 112 remaining states of the **120** adjoint representation. It is not hard to see that

$$\sum_{s \in \mathbf{120}} (Q_a^s y_6^a)^2 = \sum_{i \neq j \in \{1, \dots, 8\}} \sum_{\epsilon_1, \epsilon_2 = \pm 1} (\epsilon_1 y_6^i + \epsilon_2 y_6^j)^2. \quad (\text{A.7})$$

Therefore we get a quadratic polynomial in the y_6^i 's. Noticing that this polynomial is invariant under the transformations $y_6^i \leftrightarrow -y_6^i$ and $y_6^i \leftrightarrow y_6^j$, we obtain

$$\sum_{s \in \mathbf{120}} (Q_a^s y_6^a)^2 = \alpha \sum_{i=1}^8 (y_6^i)^2. \quad (\text{A.8})$$

Computing the $(y_6^1)^2$ term, we get $\alpha = 28$.

For the **128**, we see that

$$\sum_{s \in \mathbf{128}} (Q_a^s y_6^a)^2 = \sum_{\epsilon_1, \dots, \epsilon_7 = \pm 1} \frac{1}{4} \left(\epsilon_1 y_6^1 + \epsilon_2 y_6^2 + \dots + \left(\prod_{i=1}^7 \epsilon_i \right) y_6^8 \right)^2. \quad (\text{A.9})$$

If we set $y_6^8 = 0$, the symmetries $y_6^i \leftrightarrow -y_6^i$ and $y_6^i \leftrightarrow y_6^j$, valid for $i, j = 1, \dots, 7$, guarantee that this polynomial will be of the form

$$\beta \sum_{i=1}^7 (y_6^i)^2. \quad (\text{A.10})$$

Restoring $y_6^8 \neq 0$ gives a $(y_6^8)^2$ term and crossed terms $y_6^i y_6^8$. However, y_6^8 has been artificially isolated in the treatment of the GSO constraint: by isolating other y_6^i 's and using the same arguments, we can show that our polynomial is of the form

$$\sum_{s \in \mathbf{128}} (Q_a^s y_6^a)^2 = \beta \sum_{i=1}^8 (y_6^i)^2. \quad (\text{A.11})$$

We obtain $\beta = 32$.

It is then straightforward to evaluate the sums encountered before. We obtain

$$M_T^2 = \frac{2^3}{4\pi R_6^2} \left(60 \sum_{i=1}^{16} (y_6^i)^2 \right), \quad (\text{A.12})$$

and when coupling the Scherk-Schwarz cycle to the helicity of the E_8 representation

$$M_V^{(2)} = \frac{2^3}{4\pi R_6^2} \left(-4 \sum_{i=1}^{16} (y_6^i)^2 \right). \quad (\text{A.13})$$

In the $N = 2$ models, the orbifolding breaks $E_8 \rightarrow E_7 \times SU(2)$, under which the adjoint representation decomposes as

$$\mathbf{248} \rightarrow (\mathbf{133}, \mathbf{1}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}). \quad (\text{A.14})$$

The Cartan generators of E_8 give the Cartan generators of $E_7 \times SU(2)$:

$$(i\partial H_1, \dots, i\partial H_6, i(\partial H_7 - \partial H_8)); \quad i(\partial H_7 + \partial H_8). \quad (\text{A.15})$$

Switching on arbitrary $y_6^1, \dots, y_6^7, y_6^8$, we compute the charges of the various states step by step.

In the $\mathbf{133}$, we have 7 neutral Cartan operators, 60 ladder operators in the $Adj(SO(12))$ subalgebra

$$e^{i(\pm H_j \pm H_k)}, \quad j \neq k \in \{1, \dots, 6\}, \quad (\text{A.16})$$

2 ladders in the $Adj(SU(2))$

$$e^{\pm i(H_7 - H_8)}, \quad (\text{A.17})$$

and 64 ladders in a spinorial representation

$$e^{\frac{i}{2}(\pm H_1 \pm \dots \pm H_6 \pm (H_7 - H_8))} \quad (\text{A.18})$$

obeying a GSO condition. We see that y_6^7 has a particular role here. The latter states have charges $\pm \frac{1}{2}$ under the first six Cartan generators, and charges ± 1 under the seventh. Using the same arguments as before, we see that the sum for the spinorial states is of the form

$$\alpha \left(\sum_{i=1}^6 (y_6^i)^2 + (2y_6^7)^2 \right). \quad (\text{A.19})$$

The polynomial we are looking for is therefore

$$\sum_{s \in \mathbf{133}} (Q_a^s y_6^a)^2 = 20 \sum_{i=1}^6 (y_6^i)^2 + 2(2y_6^7)^2 + 16 \left(\sum_{i=1}^6 (y_6^i)^2 + (2y_6^7)^2 \right) = 36 \sum_{i=1}^6 (y_6^i)^2 + 72(y_6^7)^2. \quad (\text{A.20})$$

Note that if we want to couple the Scherk-Schwarz cycle to the helicity of the E_7 , we have to compute also $\sum_s \text{sign}(s)(Q_a y_6^a)^2$, where states in the spinorial representations of the $SO(12)$ subgroup contribute with a minus sign. To get this sum we have to put a minus sign in front of the **64** part, so that

$$\begin{aligned} \sum_{s \in \mathbf{133}} \text{sign}(s)(Q_a^s y_6^a)^2 &= 20 \sum_{i=1}^6 (y_6^i)^2 + 2(2y_6^7)^2 - 16 \left(\sum_{i=1}^6 (y_6^i)^2 + (2y_6^7)^2 \right) \\ &= 4 \sum_{i=1}^6 (y_6^i)^2 - 56(y_6^7)^2. \end{aligned} \quad (\text{A.21})$$

For the **(56,2)** representation, we begin with the states with vertex operators

$$e^{\pm H_i \pm H_7}, \quad e^{\pm H_i \pm H_8}. \quad (\text{A.22})$$

The corresponding $Q_a^s y_6^a$ are respectively

$$\pm y_6^i \pm (y_6^7 + y_6^8), \quad \pm y_6^i \pm (y_6^7 - y_6^8). \quad (\text{A.23})$$

Therefore the sum for these states equals

$$\begin{aligned} \sum_s (Q_a^s y_6^a)^2 &= 4 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7 + y_6^8)^2 + 4 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7 - y_6^8)^2 \\ &= 8 \sum_{i=1}^6 (y_6^i)^2 + 48 ((y_6^7)^2 + (y_6^8)^2). \end{aligned} \quad (\text{A.24})$$

The remaining states to be considered have vertex operators

$$e^{\frac{i}{2}(\pm H_1 \pm \dots \pm H_6 \pm (H_7 + H_8))}. \quad (\text{A.25})$$

For these, we get

$$16 \left(\sum_{i=1}^6 (y_6^i)^2 + (2y_6^8)^2 \right). \quad (\text{A.26})$$

Adding everything, we get the final result for the representation:

$$\sum_{s \in (\mathbf{56}, \mathbf{2})} (Q_a^s y_6^a)^2 = 24 \sum_{i=1}^6 (y_6^i)^2 + 48 (y_6^7)^2 + 112 (y_6^8)^2. \quad (\text{A.27})$$

If we couple to the E_7 helicity, we also have

$$\sum_{s \in (\mathbf{56}, \mathbf{2})} \text{sign}(s) (Q_a^s y_6^a)^2 = -8 \sum_{i=1}^6 (y_6^i)^2 + 48 (y_6^7)^2 - 16 (y_6^8)^2. \quad (\text{A.28})$$

For the $\mathbf{3}$ of $SU(2)$ the two states

$$e^{\pm i(H_7 + H_8)} \quad (\text{A.29})$$

have charges ± 2 . So we get

$$\sum_{s \in (\mathbf{1}, \mathbf{3})} (Q_a^s y_6^a)^2 = 8 (y_6^8)^2. \quad (\text{A.30})$$

In the twisted sector, we encounter the representation $(\mathbf{56}, \mathbf{1})$, whose sum is obtained by switching off y_6^8 in the result obtained for the $(\mathbf{56}, \mathbf{2})$ representation and by dividing the result by 2. One then obtains,

$$\sum_{s \in (\mathbf{56}, \mathbf{1})} (Q_a^s y_6^a)^2 = 12 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7)^2, \quad (\text{A.31})$$

$$\sum_{s \in (\mathbf{56}, \mathbf{1})} \text{sign}(s) (Q_a^s y_6^a)^2 = -4 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7)^2. \quad (\text{A.32})$$

We also encounter the $(\mathbf{1}, \mathbf{2})$ representation, where the sum equals $2(y_6^8)^2$.

Application to Models 3 and 4

For model 4, we set $Q_R = Q_F + Q_H$. If we consider Wilson lines corresponding to the 16 Cartan generators of $E_8 \times E_7 \times SU(2)$, the result is

$$M_{T,V}^{2,(2)} = \frac{4}{4\pi R_6^2} \left[60 \sum_{i=9}^{16} (y_6^i)^2 + 36 \sum_{i=1}^6 (y_6^i)^2 + 72 (y_6^7)^2 + 24 \sum_{i=1}^6 (y_6^i)^2 + 48 (y_6^7)^2 + 112 (y_6^8)^2 + 8 (y_6^8)^2 \right] \\ \pm \frac{1}{4\pi R_6^2} \left[32 \left(12 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7)^2 \right) + 128 (2(y_6^8)^2) \right].$$

So we get

$$M_T^2 = \frac{1}{4\pi R_6^2} \left(240 \sum_{i=9}^{16} (y_6^i)^2 + 624 \sum_{i=1}^6 (y_6^i)^2 + 1248 (y_6^7)^2 + 736 (y_6^8)^2 \right) \\ M_V^{(2)} = \frac{1}{4\pi R_6^2} \left(240 \sum_{i=9}^{16} (y_6^i)^2 - 144 \sum_{i=1}^6 (y_6^i)^2 - 288 (y_6^7)^2 + 224 (y_6^8)^2 \right). \quad (\text{A.33})$$

For model 3, we set $Q_R = Q_a + Q_H + Q_{E_7}$. We get the same expression for M_T^2 , while

$$\begin{aligned}
M_V^{(2)} &= \frac{4}{4\pi R_6^2} \left[4 \sum_{i=1}^6 (y_6^i)^2 - 56 (y_6^7)^2 - 8 \sum_{i=1}^6 (y_6^i)^2 + 48 (y_6^7)^2 - 16 (y_6^8)^2 + 8 (y_6^8)^2 + 60 \sum_{i=9}^{16} (y_6^i)^2 \right] \\
&\quad - \frac{1}{4\pi R_6^2} \left[32 \left(-4 \sum_{i=1}^6 (y_6^i)^2 + 24 (y_6^7)^2 \right) + 128 (2(y_6^8)^2) \right] \\
&= \frac{1}{4\pi R_6^2} \left(112 \sum_{i=1}^6 (y_6^i)^2 - 800 (y_6^7)^2 - 288 (y_6^8)^2 + 240 \sum_{i=9}^{16} (y_6^i)^2 \right). \tag{A.34}
\end{aligned}$$

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